Vortex-dynamics model for entrainment in jets and plumes

K. R. Sreenivas
Jawaharlal Nehru Center for Advanced Scientific Research, Bangalore 560064, India

Ajay K. Prasad a)
Department of Mechanical Engineering, University of Delaware, Newark, Delaware 19711

(Received 14 July 1999; accepted 4 May 2000)

Recent experimental results of Bhat and Narasimha (1996) have revealed a dramatic difference in the entrainment between jets and plumes subjected to off-source volumetric heating and their unheated counterparts. Experimental observations show that plumes entrain more rapidly than jets; the greater entrainment by the plume is typically attributed to the presence of buoyancy in the plume. In contrast, the addition of buoyancy away from the source by volumetric heating produces the opposite effect of reduced entrainment. Apart from buoyancy, other factors such as acceleration due to pressure gradients or other body forces can also affect the rate of entrainment. In this paper, we develop a model for entrainment to explain the mechanism by which buoyancy produces contrasting effects on entrainment in volumetrically heated flows in comparison to their unheated counterparts. The model highlights the role of density stratification in the process of vortex sheet roll-up in free shear flows. With this model, we are also able to explain the higher entrainment of the plume relative to the unheated jet. The model is further extended to explain entrainment behavior during acceleration due to an applied pressure gradient or other body forces. © 2000 American Institute of Physics. [S1070-6631(00)01408-2]

INTRODUCTION

Jets and plumes are unbounded flows which develop in open ambient fluid. They possess velocity gradients created by some upstream mechanism which are smoothed out by diffusion and convective deceleration. A jet is a discharge of fluid into an identical ambient fluid and is driven by its initial momentum. A plume on the other hand is driven purely by buoyancy supplied at its origin. Jets and plumes spread in a direction normal to the primary flow by the process of entrainment in which irrotational ambient fluid is incorporated into the turbulent region. The mean inward velocity of ambient fluid across the boundary of the jet or plume is known as entrainment velocity.

At a sufficient downstream distance from the origin, the dynamics of turbulent jet or plume flow are independent of initial conditions and are governed by the local velocity and length scales (self-similar region). Assuming self-similarity, Morton, Taylor, and Turner (MTT) proposed that the entrainment velocity is proportional to the local velocity scale (the Taylor hypothesis). Usually, the time-averaged center-line velocity \( U_c \) at that location is taken as the velocity scale,\(^1\)

\[
U_e = \alpha U_c. \tag{1}
\]

The entrainment coefficient, \( \alpha \), relates the local velocity scale to the entrainment velocity \( U_e \) and has to be determined by experiment. The numerical value of \( \alpha \) depends on the local velocity profile, the chosen length scale, and also on the type of flow (jet–plume in uniform/stratified environment). Conventionally, the radius at which the velocity reduces to \( 1/e \) of the center-line value is chosen as the length scale.\(^3\) The entrainment coefficient of the plume (\( \alpha_p = 0.08 \)) is found to be greater than that of the jet (\( \alpha_J = 0.05 \)).\(^2,3\) Hence, the dilution rate (rate of incorporation of irrotational ambient fluid into the turbulent flow) in a plume is higher than in a jet, for the same local momentum flux.

The MTT model with Taylor’s hypothesis of entrainment [Eq. (1)] is well-established, and is quite successful in predicting jet and plume behavior for a variety of flow conditions. The model is also applicable over a wide parametric range from lower-end laboratory-scale models (\( \text{Re}_{\sim} \text{few thousand} \)) to upper-end geophysical flows (\( \text{Re} > 10^8 \)).\(^3\) Although the MTT model does not address the variation of \( \alpha \) in different flow conditions, once this value is determined from an experiment, the model is successful in predicting the velocity and dilution rate as a function of downstream distance. The model is used to predict dispersion of pollutant plumes discharged into the atmosphere by industries and sewage discharged into oceans and rivers.

There are, however, many cases in engineering and atmospheric flows where the MTT model must be used with caution. For instance, field observations of Colorado cumuli\(^4,5\) clearly show that entrained air in a cumulus cloud originates either from the cloud base or from the cloud top, i.e., lateral entrainment is negligible. If one applies the standard plume value of \( \alpha_p = 0.08 \) to model a cloud, one can either predict the vertical rise of the cloud, or its water vapor content correctly, but not both.\(^5,6\) Atmospheric scientists have found that they can predict cloud behavior correctly with the MTT model by setting \( \alpha = 0 \) above the cloud base, indicating that ambient air is hardly mixing into the cloud as...
it ascends in the atmosphere. Narasimha’s group\(^7\)–\(^9\) has conducted a series of experiments and numerical simulations with jets and plumes subjected to volumetric heating to show that latent heat release during condensation above the cloud base is the key factor in reducing the rate of entrainment in clouds.

Apart from buoyancy, other factors that affect entrainment are flow acceleration or deceleration due to an applied pressure gradient along the flow or other body forces, and stratification of the ambient medium. Results from the reacting mixing layers\(^1\) indicate that entrainment is reduced by as much as 30% in a reacting mixing layer (flame) compared to the non-reacting mixing layer. This reduction is attributed to flow acceleration caused by dilation. Experiments of Choi et al.\(^1\) show that entrainment and mixing is high in jets subjected to an adverse pressure gradient, whereas the entrainment is reduced or completely suppressed in a jet subjected to a favorable pressure gradient.\(^1\) Entrainment is also inhibited in a jet when the injection velocity is increased in time,\(^1\) although in this instance the effect arises from temporal variations; in our paper, we limit ourselves to spatially accelerating, albeit steady (in a time-averaged sense) flows.

Figure 1 summarizes the above discussion. A jet is driven by its initial momentum supplied at the source and has an entrainment coefficient, \(\alpha_J = 0.05\). A plume, which is driven by buoyancy supplied at its source, has an entrainment coefficient \(\alpha_P = 0.08\). Therefore, at first glance, it appears as if buoyancy enhances entrainment. However, when jets and plumes are subjected to off-source volumetric heating, entrainment is reduced. Here, buoyancy has an opposite effect on entrainment. Similarly, a favorable pressure gradient, or flow acceleration, decreases the rate of entrainment, whereas an adverse pressure gradient, or flow deceleration, increases entrainment. In modeling flows subjected to the various conditions described above, it is important to understand the mechanism by which these factors influence the process of entrainment. This will be useful in modeling many atmospheric and engineering flows. Ideally, any pro-
posed mechanism should universally explain how entrainment gets affected by all of the above factors.

In the literature, we find different models to explain/parametrize entrainment in jets, plumes, and buoyant jets. For example, Priestly and Ball,\textsuperscript{14} relate the entrainment coefficient to the local Richardson number ($\text{Ri}$) as follows:

$$\alpha = \alpha_J - (\alpha_J - \alpha_P) \left( \frac{\text{Ri}}{0.557} \right)^2,$$

where $\alpha$ is the local entrainment coefficient, $\alpha_J = 0.05$ and $\alpha_P = 0.08$. Note that this parametrization fails to predict the lower entrainment observed in volumetrically heated jets and plumes.

Hunt,\textsuperscript{15} based on physical scaling, has explained the higher entrainment of the plume compared to that of the jet by arguing that any body force present in the direction of the flow would increase entrainment; again, Hunt’s model does not explain the lower entrainment in volumetrically heated jets. Similarly, Lumley\textsuperscript{16} explains the higher degree of mixing in the plume by comparing the eddy viscosities in jets and plumes. According to him, a jet is subjected to a transverse straining field due to the entrainment velocity which compresses the eddies laterally resulting in a lower eddy viscosity. In contrast, a plume is able to counteract this lateral compression by a lateral expansion\textsuperscript{16} arising from the non-zero axial temperature gradient. Once again, Lumley’s\textsuperscript{16} theory predicts higher rates of mixing for a volumetrically heated jet compared with an unheated jet, which is contrary to experimental observations.

Bhat and Narasimha\textsuperscript{7} offer an explanation for the reduced entrainment in a volumetrically heated jet by noting the effect that buoyancy produces on the large coherent structures residing in the jet. They argue that the flow at the edge of the jet is moving more slowly than the central core. Consequently, the residence time of the outer fluid in the heating zone is greater, resulting in a greater temperature rise, and therefore buoyancy, at the edges as compared to the core. They conclude that such a radially varying buoyancy addition will “disrupt the prevailing coherent structure” and diminish entrainment by engulfment.\textsuperscript{7} However, Bhat and Narasimha\textsuperscript{7} do not describe the precise mechanism by which such disruption reduces entrainment. Furthermore, such an argument cannot be applied to explain the greater entrainment of an ordinary plume in comparison to the ordinary jet.

One may conclude, therefore, that there does not currently exist a model which can satisfactorily explain all of the observed entrainment behaviors in jets and plumes. In the next section we describe a physical model for the process of entrainment and explain how the various factors discussed earlier can affect entrainment.

**MECHANISM OF ENTRAINMENT AND VOLUMETRIC HEATING**

Near the origin of a shear flow, a vortex sheet separates the jet–plume fluid containing momentum from the stationary ambient fluid. Further downstream, this vortex sheet being unstable, rolls up into a sequence of discrete vortices. Vortex sheet roll-up is an instability process and the separation between these vortex structures (eddies) is proportional to the local width of the flow and the velocity profile.\textsuperscript{17} Subsequently, these eddies interact by rolling around each other due to their mutually induced velocities. As these eddies roll around, they entrap or engulf irrotational ambient fluid and incorporate it into the turbulent flow, as shown in Fig. 2.

---

**FIG. 2.** Axisymmetric jet in the near field; shaded region indicates entrapped ambient fluid during the process of rolling around.

**FIG. 3.** Jet–plume in the far field; (a) instantaneous entrained fluid motion with an eddy and instantaneous interface; (b) time-averaged entrained fluid motion and interface. From Ref. 26 with permission.
This first step in the process of entrainment is kinematic and is known as the *induction phase*.\(^\text{18}\) The inducted fluid, although still irrotational, forms a part of the moving turbulent fluid. Subsequent to the induction phase, Dimotakis\(^\text{18}\) describes two additional steps to complete the entrainment process. Turbulent straining of the inducted fluid reduces its spatial scale to a small enough value at which viscous diffusion dominates (diastrophy). Finally, due to viscous diffusion, the inducted fluid is mixed at the molecular level with the turbulent flow (infusion). From the above description, it is apparent that the key step in the entrainment process is the one that controls the rate at which ambient fluid enters into the turbulent region, i.e., the induction phase.

Such eddy formation and rolling around is evident in the near- and far-field of a mixing layer.\(^\text{19,20}\) The process of rolling around of eddies in mixing layers leads to the coalescence of eddies and the formation of a single larger eddy. Similar eddies are seen both in the near- and far-field of jets and plumes,\(^\text{9,21–24}\) however, the mechanism of rolling of eddies around each other is seen only in the near field.\(^\text{22,24,25}\) Evidence for eddies rolling around and coalescing in the far field of a buoyant jet is weak.\(^\text{26}\) Thus the mechanism of induction in jets and plumes could arise from a single eddy structure as suggested by Mungal;\(^\text{26}\) see Fig. 3. Therefore, while mixing layers entrain by both (a) sweeping in fluid by the induced velocity, and (b) physically entrapping the irrotational ambient fluid during rolling around of eddies, only the first mechanism is ostensibly operational in jets and plumes. In either case (i.e., whether rolling around is present or not), entrainment strongly depends on the induction velocity produced by the eddy structures observed in free shear flows. Any mechanism that interferes with the induction process will also affect entrainment.

Consider now, two fluid elements \(A\) and \(B\), which belong to an eddy of size \(\lambda\) on the interface between the jet and the ambient medium, Fig. 4(a). (We use the jet as the model flow for the analysis; subsequently, the analysis will be extended to plumes.) Let the eddy rotate with an angular velocity \(\omega_{AB}\). We can estimate \(\omega_{AB}\) by assuming that (a) the vorticity is distributed uniformly in the eddy, and (b) the total circulation, \(\Gamma\), contained in the eddy is proportional to the centerline velocity and the size of the eddy, \(\lambda\). That is

\[
\Gamma = C_1 U_c \lambda, \tag{2}
\]

where \(C_1\) is a constant. \(\omega_{AB}\) may be related to \(U_c\) and \(\lambda\) according to
\[ \omega_{AB} = \frac{2 \Gamma}{\pi \lambda^2} = \frac{2C_1 \, U_c}{\pi \lambda}. \]  

(3)

Finally,

\[ V_A = V_B = \frac{\omega_{AB} \lambda}{2}. \]  

(4)

where \(V_A\) and \(V_B\) are the induced velocities at A and B, respectively due to the vorticity in the eddy. Now, in a jet subjected to volumetric heating [Fig. 4(b)], the parcel of fluid at B has resided longer in the heating zone than the parcel at A, hence the parcel at B will have a higher temperature than the fluid parcel at A. The lighter fluid at B, due to its higher temperature, is now able to resist the downward displacement caused by induced velocity, i.e., the stable stratification caused by volumetric heating produces a restoring torque which opposes the torque caused by the eddy vorticity. As the following analysis shows, it is possible to express the relative magnitudes of these torques as a dimensionless parameter which predicts entrainment behavior in the presence of off-source heating.

The angular velocity, \(\omega_T\), of the restoring baroclinic torque is proportional to \(\sqrt{g \beta_T T_e}\) (which is the Brunt–Väisälä frequency). Here, \(\beta_T\) is the thermal expansion coefficient, and \(T_e\) is the rate of temperature rise in the vertical direction due to the volumetric heating. We can further write

\[ \omega_T = C_2 \sqrt{g \beta_T T_e} = C_2 \frac{g \beta_T T_e}{U_c \rho c_p} \left( \frac{q^m}{U^2 c_p} \right)^{1/2}, \]  

(5)

where \(q^m\) is the rate of heat addition per unit volume of the jet, \(\rho\) is the density, \(c_p\) is the specific heat, and \(C_2\) is a constant. Induction can proceed in a volumetrically heated jet only if the angular velocity due to eddy-vorticity, \(\omega_{AB}\), is greater than the restoring angular velocity due to buoyancy, \(\omega_T\). Therefore, entrainment will continue when \(\omega_T\) opposes \(\omega_{AB}\) only if \(|\omega_{AB}| > |\omega_T|\). From Eqs. (3) and (5), this condition may be written as

\[ \frac{\lambda}{b} < \frac{C_1}{C_2} \frac{2}{\pi} \left( \frac{U_c^3 \rho c_p}{b^2 g \beta_T q^m} \right)^{1/2}, \]  

(6)

where \(b\) is the local length scale. Further

\[ \frac{\lambda}{b} < \frac{C_1}{C_2} \frac{2}{\pi} \left( \frac{U_c^3 \rho c_p}{g \beta_T q^m} \right)^{1/2}, \]  

(6)

where \(q^m\) is the rate of heat addition per unit length of the jet. The resulting angular velocity of the eddy, \(\omega_R\), and eddy turnover time, \(t_e\), are given below

\[ \omega_R = \omega_{AB} + \omega_T, \quad t_e = \frac{1}{\omega_R}. \]  

(7)

The result contained in (6) places an upper bound on the size of eddy which can continue to rotate in the presence of volumetric heating. Due to the restoring baroclinic torque, eddy rotation and hence entrainment will be arrested for eddies which have \(\lambda/b\) larger than that given by (6). For eddies smaller than this critical value, rotation will continue with a reduced angular velocity \(\omega_R\) and hence a larger eddy turnover time \(t_e\). Thus, the effect of volumetric heating is to (a) reduce \(\lambda\), and (b) increase \(t_e\). Both of these factors will reduce entrainment in the case of volumetrically heated flows. Thus, large scale engulfment is suppressed and only small scale “nibbling” at the edge of the jet is possible. \(\lambda/b\) is then the desired dimensionless parameter which governs the entrainment process.

It should be noted that stable stratification caused by volumetric heating will retard entrainment irrespective of the angle made by line \(AB\) with the jet-axis. Accordingly, entrainment will reduce when the jet diverges [as shown in Fig. 4(b)], or when it narrows at high rates of off-source heating, or when it rises vertically without lateral spreading for moderate rates of off-source heating. Essentially, the rotation under the Biot–Savart induced torque is a prerequisite for the engulfment phase of entrainment, and stable stratification provides a torque that resists such a rotation under all conditions. It should also be noted that result (6) was derived without specifying whether line \(AB\) is parallel to, diverges from, or converges towards the jet axis. It is applicable in all three scenarios.

The relative magnitudes of \(\omega_{AB}\) and \(\omega_T\) may be estimated from Ref. 7. Using their results we see that \(U_c \approx 4 \text{ cm/s}, \lambda \approx 4 \text{ cm},\) and \(T_e \approx 0.2 \text{ °C/cm} \) (at moderate heating rates). Then, from Eqs. (3) and (5), respectively, \(\omega_{AB} = 0.636 C_1\), and \(\omega_T = 0.25 C_2\). Given that \(C_1\) and \(C_2\) are of order unity, these estimates show that both processes have similar magnitudes and that their interaction could affect the entrainment process.

It could be argued from result (6) that because \(U_c\) increases under the influence of volumetric heating, so also would \(\lambda/b\) increase, leading to an increase in entrainment which contradicts our model. However, the observations reported in Ref. 7 indicate that the increase in \(U_c\) is quite small, especially when compared to the increase in \(q^m\) required to bring about that change. Therefore, the rather small increase in \(U_c\) is overwhelmingly countered by stable stratification arising from volumetric heating and result (6) is still valid.

The numerical results of Basu and Narasimha indicate that the local, absolute value of vorticity in a heated jet can increase ten-fold over an unheated jet. One might therefore argue that such an increase in vorticity due to volumetric heating might actually cause an increase in entrainment. However, according to our model, entrainment depends not so much on the local vorticity, but the overall circulation which increases only marginally. Essentially, while both positive and negative components of vorticity increase substantially, their effects cancel out while computing the induced velocity over a length scale comparable to the width of the jet or plume. The rolling over of a large coherent structure is determined more by this net circulation and less by the smaller-scale structure (with oppositely signed albeit more vigorous vorticity existing in close proximity). Consequently, the substantial increase in entrophy does not lead to a corresponding increase in net circulation, and entrainment is more directly affected by the stable stratification resulting from the vertical temperature gradient.

The result contained in (6) can be related to the prevail-
ing axial temperature gradient $T_z$ and used to compare the relative entrainment in a number of situations of interest: (1) volumetrically heated jets with ordinary jets; (2) ordinary jets with plumes; (3) volumetrically heated plumes with ordinary plumes; and (4) jets and plumes subjected to acceleration or deceleration due to pressure gradients or other body forces.

(1) Ordinary jets [Fig. 4(a)] do not contain any temperature effects, therefore, the axial temperature gradient $T_z$ is zero, and $\omega_T$ is also zero. In contrast, the volumetrically heated jet [Fig. 4(b)] possesses a positive $T_z$ because the jet fluid is continuously heated as it rises through the heating zone. The existence of this positive $T_z$ imposes a baroclinic torque which opposes the torque imposed by the vorticity contained in the eddy, i.e., $\omega_T$ has an opposite sign to $\omega_{AB}$, therefore, the resulting angular velocity $\omega_R$ is reduced, and eddy turnover time $t_e$ is increased. Therefore, eddies in a volumetrically heated jet should turn around more slowly and entrain less vigorously than in an ordinary jet.

(2) Ordinary plumes are driven purely by the buoyancy introduced at the source. As the plume rises vertically, the temperature decreases in the axial direction due to entrainment and diffusion according to $T \sim z^{-5/3}$. The associated temperature gradient $\frac{dT}{dz}$ is negative, and the downstream part of an eddy is, on average, cooler and denser than the upstream part, resulting in an unstable density stratification [see Fig. 4(c)]. Consequently, the fluid will tend to overturn when perturbed. The perturbation in this case is supplied by the Biot–Savart induced velocity $\omega_{AB}$. In other words, the baroclinic torque aids the torque due to eddy vorticity, and $\omega_T$ acts in the same direction as $\omega_{AB}$, giving a greater $\omega_R$, and smaller $t_e$. The potential energy stored in unstable stratification is thus released, and consequently, plumes entrain more rapidly than jets.

(3) The addition of off-source buoyancy by volumetric heating in a plume results in a $T_z$ which could be positive or negative, depending on the rate of heat addition, and the rate at which the axial temperature would have dropped in the plume in the absence of heating. In either case, the presence of volumetric heating serves to make $T_z$ less negative in comparison to the ordinary plume. Consequently, $\omega_T$, which would have been positive (i.e., in the same direction as $\omega_{AB}$) in the absence of volumetric heating, now becomes less positive or even negative with the addition of volumetric heating. The resulting $\omega_R$ decreases, $t_e$ increases, and entrainment in a volumetrically heated plume is less than that in an ordinary plume.

(4) Consider now a jet–plume subjected to an axial acceleration due to a favorable pressure gradient, or dilution by combustion, or other body force such as a magnetic field. In all these cases, the fluid parcel at B has resided longer in the acceleration zone, and so will have higher velocity than the parcel at A [Fig. 4(d)]. The resulting torque due to acceleration will oppose the downward pull due to eddy vorticity. Hence the effect of axial acceleration is similar to volumetric heating, and entrainment will be reduced. Conversely, an axial deceleration will reduce the eddy turn over time and aid the process of entrainment.

The influence of acceleration is certainly present even in the case of off-source buoyancy addition by volumetric heating, and the combined effects of buoyancy and acceleration are consistent with each other. In fact, it is possible to extend the argument by noting that sufficiently large accelerations can eventually lead to relaminarization with the concomitant reduction in entrainment.

Table I summarizes the above discussion.

While the above analysis considers an isolated vortex, in reality, there will exist many such eddies arrayed along the shear layer with long-range interactions between them. We conducted simulations to investigate the effect of a line of such vortices on the critical eddy size $\lambda/b$ and the resultant angular velocity $\omega_R$. In our simulations, the vortex-centers were initially located along the same line, and subsequently, alternate vortices were shifted to one side of the line, while the others were shifted to the other side. The amount of sideways shift was systematically increased (to a maximum of $90^\circ$) and the effect of this vortex array was studied on one vortex located towards the middle of the array. Results show some sensitivity to the exact orientation of the eddy centers, however, due to cancellation of the effects of vortices located on opposite ends of the array the overall effect of multiple vortices is of order unity (in the extreme case where the sideways shift was maximized, $\omega_{AB}$ was multiplied by a factor of 2.5).

A possible shortcoming of the model is that it is two-dimensional, whereas the flow structures in free-shear layers are three-dimensional in nature. Despite this limitation, the model is able to predict (at least qualitatively) the entrainment behavior over a wide variety of free-shear flows in a unified manner.

**CONCLUSIONS**

The proposed model explains the anomalous effect of buoyancy on the entrainment process in volumetrically heated jets and plumes. Essentially, the addition of off-source buoyancy creates a stable density stratification which resists the tendency of eddies to turn over under the influence of their vorticity. Consequently, volumetrically heated flows experience reduced entrainment in comparison to their unheated counterparts. In the case of an ordinary plume wherein the addition of buoyancy is confined to the source...
(before entering into the ambient medium), decreasing buoyancy along the downstream direction results in an unstable density stratification and a baroclinic torque which aids the vorticity-induced velocity. Hence, plumes entrain more than jets which do not contain density variations. Similarly, this analysis can be extended to a flow with favorable pressure gradient to explain the decrease (increase) in entrainment.

ACKNOWLEDGMENTS

This work was made possible by a grant from NSF, Division of Atmospheric Sciences (ATM-9714810). K.R.S. thanks Professors Narasimha and Bhat for introducing him to this problem. We thank Professor Narasimha for several useful discussions, and Professor A. S. Wexler for educating us about cloud thermodynamics.
