Local Input-to-State Stability of Dynamic Walking Under Persistent External Excitation using Hybrid Zero Dynamics

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***Abstract*— This paper establishes local input-to-state stability (ISS) of a dynamically (limit-cycle) walking biped under the effect of persistent exogenous forcing. For applications involving interaction of a walking biped with an external agent, the biped should be able to adapt its locomotion to external forces. Local ISS guarantees a bound on the magnitude of the exogenous force within which the biped will continue taking steps. On a point foot biped, a controller is developed within the framework of hybrid zero dynamics (HZD) to generate an exponentially stable walking gait in the absence of the external force. The corresponding HZD is shown to be locally ISS when an external piecewise constant force acts on the biped. Local ISS for the full-order biped is then proved under the assumption of sufficiently fast convergence rates of the transversal dynamics. These results provide a first step toward a rigorous framework within which tasks that involve dynamic locomotion under the influence of external forcing can be analyzed.**

I. Introduction

Collaboration of a dynamically (limit-cycle) walking bipedal robot with an external agent whose intentions are not explicitly known to the robot calls for gait adaptation based on the interaction force, while simultaneously ensuring that the biped does not fall. This necessitates the design and analysis of controllers that can harness the interaction force for gait adaptation without destabilizing the robot. In the presence of an exogenous force, the notion of exponential stability does not guarantee good behavior with respect to the external force. Instead, a broader notion of stability – namely, input-to-state stability (ISS) – is needed to ensure that the biped adapts its walking gait to external activity. This paper establishes local ISS of a dynamically walking biped that adjusts its speed in accordance to the external force.

Quasi-static bipedal robots have a rich literature of controllers that can handle manipulation and cooperation tasks. A variety of Zero Moment Point (ZMP) based methods can be found in [1] for tasks involving exogenous forces. Dynamically stable bipeds still lack in this regard. While various methods are available for generating stable periodic orbits, these methods are developed in the absence of any external force. One such method is the hybrid zero dynamics (HZD) [2], and its recent extensions [3]–[5]. There are very few controllers available in the dynamic walking literature that are designed explicitly taking into account external forces. Recent work in [6] adopts a null-space control approach so that the forces external to the locomotion system do not interfere with it; this approach is suitable for unifying manipulation with locomotion tasks. On the other hand, [7], [8] use the interaction force explicitly for speed adaptation and will be of particular interest here.

The ISS property has proved useful in stability analysis of continuous-time nonlinear systems [9], [10] as well as discrete-time systems [11]. Intuitively, an ISS system exhibits bounded state trajectories in response to bounded inputs and the trajectories converge to the nominal motion as the inputs tend to zero [10]. In our setting, we treat the external force as an input to the closed loop system and we formally establish the local ISS property.

We adopt a fairly generic planar model of an underactuated biped that is persistently under external forcing. Then, a HZD controller is developed to generate an exponentially stable periodic motion in the absence of the external force. When the force is applied, the underactuated biped responds to the force by favorably changing its speed. In this setting, the notion of ISS emerges naturally as a suitable stability concept that treats the external force as an input to the biped and ensures that it does not destabilize its motion. We establish local ISS of the full order Poincaré map in two stages. First we propose an ISS Lyapunov function that renders the restricted to the zero dynamics manifold Poincaré map locally ISS. This stage is facilitated by the availability of the analytical form of HZD. Then, motivated by the constructions in [3], we propose a candidate for an ISS Lyapunov function, which is used to establish local ISS for the full-order Poincaré map, provided that the convergence rate of the transversal dynamics is fast enough. Our goal is to provide a framework for rigorously analyzing tasks that involve physical interaction between a walking biped and a (robotic or a human) co-worker; see [8] for an example. This work represents a first step towards such a framework.

This paper is organized as follows. Section II describes the model and the controller. Section III states our main result on input-to-state stability, and Section IV provides a proof. Section V provides numerical evidence validating the results presented in Section III. Section VI concludes the paper.

II. Modeling and Control

The bipedal model considered here comprises five links – a torso and two segmented legs each composed of a thigh and a shin, as in Fig. 1. The model has four actuators – two for each leg, actuating its hip and knee joints. A complete step of the biped consists of a swing phase and an instantaneous double stance phase. During the swing phase, the stance toe acts as a passive pivot between the shin of the stance leg and the ground. It is assumed that the stance toe does not slip

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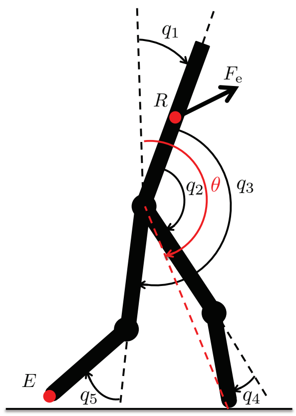

Fig. 1. Robot model with a choice of generalized coordinates.

when in contact with the ground. The mechanical properties of the model can be found in [2, Table 6.3].

A. Open-loop Hybrid Model

The configuration space Q of the model is a subset of $[0, 2\pi)^5$ containing configurations of the biped that can be physically realized. Let $q := (q_1, .., q_5)^{\mathrm{T}}$ be a set of coordinates on Q, selected to be the relative joint angles shown in Fig. 1. The swing phase dynamics are written as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu + J^{\mathrm{T}}(q)F_{\mathrm{e}} \ , \qquad (1)$$

where D is the inertia matrix, and $C\dot{q}$ and G contain velocity- and configuration-dependent forces, respectively. The input u contains the actuator torques applied at the hip and knee joints, and are mapped to the generalized forces through the matrix B. Finally, $J(q) := \partial p_{\mathrm{R}}(q)/\partial q$, with $p_{\mathrm{R}}(q)$ being the position of the point R on the torso where the exogenous force F_{e} acts; see Fig. 1. Choosing $x := (q^{\mathrm{T}}, \dot{q}^{\mathrm{T}})^{\mathrm{T}}$ as the state, (1) takes the form

$$\dot{x} = f(x) + g(x)u + g_{\mathrm{e}}(x)F_{\mathrm{e}} \ , \qquad (2)$$

where $x \in TQ := \left\{(q^{\mathrm{T}}, \dot{q}^{\mathrm{T}})^{\mathrm{T}} \mid q \in Q,\ \dot{q} \in \mathbb{R}^5\right\}$ and the vector fields f, g, and g_{e} are defined accordingly.

The swing phase is terminated when the toe of the swing leg lands on the ground; that is, when the solution of (2) intersects the switching surface

$$\mathcal{S} := \left\{(q^{\mathrm{T}}, \dot{q}^{\mathrm{T}})^{\mathrm{T}} \in TQ \mid p^{\mathrm{v}}_{\mathrm{E}}(q) = 0,\ \dot{p}^{\mathrm{v}}_{\mathrm{E}}(q,\dot{q}) < 0\right\} \ , \qquad (3)$$

where $p^{\mathrm{v}}_{\mathrm{E}}$ is the height of the toe E of the swing leg as shown in Fig. 1. Following [12, Section II-B], the double support phase is modeled by the map $\Delta : \mathcal{S} \to TQ$ taking the states just before impact x^- to states just after impact x^+; i.e.,

$$x^+ = \Delta(x^-) \ . \qquad (4)$$

Combining the swing phase (2) and the instantaneous double support phase (4), the open-loop model can be expressed in the form

$$\Sigma_{\mathrm{o}} : \begin{cases} \dot{x} = f(x) + g(x)u + g_{\mathrm{e}}(x)F_{\mathrm{e}}, & x \notin \mathcal{S} \\ x^+ = \Delta(x^-), & x \in \mathcal{S} \end{cases} \ .$$

B. Closed-loop Hybrid Model

The controller used in this paper is developed within the Hybrid Zero Dynamics (HZD) framework [12] with the assumption that the external force F_{e} can be measured. To the swing phase dynamics (2) we associate the outputs

$$y = h(q) := q_{\mathrm{c}} - h_{\mathrm{des}} \circ \theta(q) \ , \qquad (5)$$

where $q_{\mathrm{c}} = (q_2, q_3, q_4, q_5)^{\mathrm{T}}$ are the controlled variables; see Fig. 1. The desired evolution $h_{\mathrm{des}} \circ \theta(q)$ in (5) is parameterized with respect to the absolute angle of the line connecting the stance toe to the hip joint, $\theta(q) = q_1 + q_2 + \frac{1}{2}q_4$; see Fig. 1. For details on how to design h_{des} see [12].

Differentiating the output (5) twice with respect to time

$$\ddot{y} = L_f^2 h(x) + L_g L_f h(x) u + L_{g_{\mathrm{e}}} L_f h(x) F_{\mathrm{e}} \ ,$$

where $L_f^2 h$, $L_g L_f h$ and $L_{g_{\mathrm{e}}} L_f h$ are Lie derivatives of h along the corresponding vector fields; see [2, Appendix B.1.5] for definitions. Then, under the invertibility of the decoupling matrix $L_g L_f h$ and the knowledge of external force F_{e}, the input

$$u^*(x, F_{\mathrm{e}}) = -L_g L_f h^{-1}(x)\left[L_f^2 h(x) + L_{g_{\mathrm{e}}} L_f h(x) F_{\mathrm{e}}\right]$$

renders the zero dynamics surface

$$\mathcal{Z} := \{(q,\dot{q}) \in TQ \mid h(q) = 0,\ L_f h(q,\dot{q}) = 0\}$$

invariant under the swing phase dynamics. If, in addition to invariance in continuous time, the output $h_{\mathrm{des}} \circ \theta(q)$ in (5) is designed according to [2, Section 6.2] then $\Delta(\mathcal{S} \cap \mathcal{Z}) \subset \mathcal{Z}$; that is, $\mathcal{Z}$ is rendered hybrid invariant. An important observation made in [7, Section II-B] is that hybrid invariance of $\mathcal{Z}$ is preserved under the action of the external force F_{e}, provided that the force is available for feedback.

Attractivity of $\mathcal{Z}$ can be achieved by introducing an auxiliary term $\nu(y, \dot{y})$ in the control input u^* so that[1]

$$u = -L_g L_f h^{-1}\left[L_f^2 h + L_{g_{\mathrm{e}}} L_f h F_{\mathrm{e}} + \nu\right] \ , \qquad (6)$$

with the objective of ν being to drive the output (5) to zero. A number of different controllers can be used for ν; see [3] for different possibilities. In this paper,

$$\nu(y, \dot{y}) = \frac{1}{\epsilon^2} K_{\mathrm{P}} y + \frac{1}{\epsilon} K_{\mathrm{D}} \dot{y} \ , \qquad (7)$$

where K_{P} and K_{D} are constant matrices and $0 < \epsilon < 1$.

Substituting (6)-(7) in Σ_{o} results in the closed-loop dynamics of the system. It can be shown that, under the coordinates

$$\eta := \begin{bmatrix} \frac{1}{\epsilon} h(q)^{\mathrm{T}} & L_f h(q,\dot{q})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \quad \xi := \begin{bmatrix} \theta & \zeta \end{bmatrix}^{\mathrm{T}} \ , \qquad (8)$$

where $\zeta := \frac{1}{2}(D_1(q)\dot{q})^2$ and D_1 is the first row of D in (1), the closed-loop hybrid system takes the form

$$\Sigma_{\mathrm{c}} : \begin{cases} \epsilon\dot{\eta} = A\eta & \text{if } (\eta,\xi) \notin \mathcal{S} \\ \dot{\xi} = f_\xi(\eta,\xi) + g_\xi(\eta,\xi)F_{\mathrm{e}} & \\ \eta^+ = \Delta_\eta(\eta^-,\xi^-) & \text{if } (\eta^-,\xi^-) \in \mathcal{S} \\ \xi^+ = \Delta_\xi(\eta^-,\xi^-) & \end{cases} \qquad (9)$$

[1] Functional dependence has been dropped to avoid cluttering.

where $A = \begin{bmatrix} 0 & I \\ -K_{\rm P} & -K_{\rm D} \end{bmatrix}$ is Hurwitz.

Remark 1: The closed-loop vector fields f_ξ, g_ξ, and the mapping $\Delta = [\Delta_\eta^{\rm T}\ \Delta_\xi^{\rm T}]^{\rm T}$ are locally Lipschitz, and let L_f, L_g, L_Δ, L_{Δ_η}, L_{Δ_ξ} be the corresponding Lipschitz constants.

C. Full-order and reduced-order forced Poincaré maps

To keep the technical prerequisites associated with the definition of the *forced* Poincaré map at a minimum [13, Section 3.3.2], we assume that the magnitude and direction of the external force $F_{\rm e}$ are allowed to change only at transitions from one step to the next. Hence, we focus our analysis on the class $\mathcal{F}$ of piecewise constant, right continuous functions that are bounded; namely,

$$\mathcal{F} := \Big\{ F_{\rm e} \colon \bigcup_{k=0}^{\infty} [t_k, t_{k+1}) \to \mathbb{R}^2 \mid F_{\rm e}(t) = F_k \text{ for all } t \in [t_k, t_{k+1}) \text{ and } F_{\sup} < \bar{F} \Big\} ,$$

where $k \in \mathbb{Z}_+$, $\bar{F} \in \mathbb{R}_+$, and[2] $F_{\sup} := \sup_{k\in\mathbb{Z}_+} \|F_k\|$.

Let $(\eta, \xi) \in \mathcal{S}$ and let $\varphi^\epsilon(t, \Delta(\eta,\xi), F_k)$, $t \geq t_k$, denote a maximal (forced) solution based on the initial conditions $\Delta(\eta,\xi) \in TQ$ of the continuous-time part of $\Sigma_{\rm c}$ given by (9). The time-to-impact function $T_I^\epsilon : TQ \times \mathbb{R}^2 \to \mathbb{R}_+$ can then be defined as

$$T_I^\epsilon(\eta,\xi,F_k) := \inf\{t \geq t_k \mid \varphi^\epsilon(t,\Delta(\eta,\xi),F_k) \in \mathcal{S}\} . \quad (10)$$

Remark 2: The fact that (10) is well defined follows from the implicit function theorem in view of the fact that the unforced time-to-impact function $T_I^\epsilon(\eta,\xi,0)$ is well defined [12, Section II-C]. Let $(\eta,\xi) \in \mathcal{S}$ be some initial conditions for which $T = T_I^\epsilon(\eta,\xi,0) < \infty$ and define $H(t,\eta,\xi,F_k) := p_{\rm E}^{\rm v}(\varphi^\epsilon(t,\Delta(\eta,\xi),F_k))$. From the definition of $\mathcal{S}$ by (3), $H(T,\eta,\xi,0) = 0$ and $\frac{\partial H}{\partial t}|_{(T,\eta,\xi,0)} < 0$. Then, by the implicit function theorem, there exists $\bar{F} > 0$ such that for each[3] $F_k \in B_{\bar{F}}(0)$, the forced time-to-impact function (10) is well defined. Furthermore, by [14, Theorem 1.1], $T_I^\epsilon(\eta,\xi,F_k)$ is locally Lipschitz.

To define the Poincaré map, the surface $\mathcal{S}$ given by (3) is taken as the Poincaré section. Then, the full-order (forced) Poincaré map $\tilde{P}^\epsilon : \mathcal{S} \times \mathbb{R}^2 \to \mathcal{S}$ is defined by

$$\tilde{P}^\epsilon(\eta,\xi,F_k) := \varphi^\epsilon(T_I^\epsilon(\eta,\xi,F_k),\Delta(\eta,\xi),F_k) \ . \quad (11)$$

Unforced periodic walking motions correspond to fixed points $(\eta^*,\xi^*) \in \mathcal{S}$ of $\tilde{P}^\epsilon$; that is, $\tilde{P}^\epsilon(\eta^*,\xi^*,0) = (\eta^*,\xi^*)$. Although (η,ξ) is ten-dimensional, the fact that $(\eta,\xi) \in \mathcal{S}$ in the definition of $\tilde{P}^\epsilon$ restricts the dimension of the system by one. This dimensional reduction is inherent in the method of Poincaré. Furthermore, the implicit function theorem guarantees that, locally, in a neighborhood of a fixed point, the nine-dimensional vector (η,ζ) is a valid set of coordinates. As a result, the evolution of the system as it crosses $\mathcal{S}$ can be described by the discrete-time system

$$\begin{bmatrix} \eta_{k+1} \\ \zeta_{k+1} \end{bmatrix} = \begin{bmatrix} P_\eta^\epsilon(\eta_k,\zeta_k,F_k) \\ P_\zeta^\epsilon(\eta_k,\zeta_k,F_k) \end{bmatrix} =: P^\epsilon(\eta_k,\zeta_k,F_k). \quad (12)$$

[2]Notation: $\|\cdot\|$ denotes the Euclidean norm.

[3]Notation: $B_\delta(\mathrm{x})$ denotes the open ball of radius δ around x.

The restriction of the Poincaré map P^ϵ on $\mathcal{Z}$ results in a reduced-order (one-dimensional) forced Poincaré map $\rho := P^\epsilon|_{\mathcal{Z}} : (\mathcal{S} \cap \mathcal{Z}) \times \mathbb{R} \to \mathcal{S} \cap \mathcal{Z}$, which can be computed analytically as in [7, Section III] and is given below for subsequent use

$$\rho(\zeta, w_k) := \delta_{\rm z}^2 \zeta - v + w_k \ , \quad (13)$$

where ζ is defined in (8), $\delta_{\rm z}$ is a constant, and

$$v := -\int_{\theta^+}^{\theta^-} \frac{\kappa_2(\xi)}{\kappa_1(\xi)} d\xi \ ,$$

$$w_k := \int_{\theta^+}^{\theta^-} \frac{1}{\kappa_1(\xi)} \left(\kappa_3(\xi) F_k\right) d\xi \ ,$$

where θ^+ and θ^- are the touchdown and liftoff values of the angle that connects the toe of the stance leg with the hip shown in Fig. 1, and κ_1, κ_2, κ_3 are functions defined in [7, Lemma 1]; the epressions are omitted for brevity. Notice that the existence of the external forcing F_k emerges in (13) via the term w_k, which intuitively represents the "work" done by the force along a solution restricted to $\mathcal{Z}$; see [7]. As in the full-order case, the reduced-order Poincaré map (13) gives rise to the forced discrete-time dynamical system

$$\zeta_{k+1} = \rho(\zeta_k, w_k) \ , \quad (14)$$

which will be used in the following section to study the response of the system to the external force.

III. Main Result: Local Input-to-state Stability

Our objective is to understand the dependence of state trajectories of (12) on the magnitude of an externally applied force. We are concerned with discrete-time nonlinear dynamical systems which have the general form

$$\mathrm{x}_{k+1} = \mathbf{f}(\mathrm{x}_k, \mathrm{u}_k) \ , \quad (15)$$

where $\mathrm{x}_k \in \mathbb{R}^n$ and $\mathrm{u}_k \in \mathbb{R}^m$ are values of the state and input variables at the k-th discrete time respectively. Let x^* be an equilibrium point of the 0-input system; i.e., $\mathbf{f}(\mathrm{x}^*, 0) = \mathrm{x}^*$. The following definitions are adapted from [11].

Definition 1: The system (15) is locally input-to-state stable (LISS), if for all $\mathrm{x}_0 \in B_\delta(\mathrm{x}^*)$ there exists a class-$\mathcal{KL}$ function $\beta : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ and a class-$\mathcal{K}$ function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each input $\mathrm{u} \in \ell_\infty^m$,

$$\|\mathrm{x}_k - \mathrm{x}^*\| \leq \beta(\|\mathrm{x}_0 - \mathrm{x}^*\|, k) + \alpha(\sup_{k\in\mathbb{Z}_+} \|\mathrm{u}_k\|)$$

Definition 2: A continuous positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a local input-to-state (LISS) Lyapunov function if, for all $\mathrm{x}_k \in B_\delta(\mathrm{x}^*)$ and for all $\mathrm{u} \in \ell_\infty^m$, the following conditions are satisfied

$$\alpha_1(\|\mathrm{x}_k - \mathrm{x}^*\|) \leq V(\mathrm{x}_k) \leq \alpha_2(\|\mathrm{x}_k - \mathrm{x}^*\|) \quad (16)$$

$$V(\mathrm{x}_{k+1}) - V(\mathrm{x}_k) \leq -\alpha_3(\|\mathrm{x}_k - \mathrm{x}^*\|) + \alpha_4(\sup_{k\in\mathbb{Z}_+} \|\mathrm{u}_k\|) \quad (17)$$

where α_1, α_2, α_3 and α_4 are class-$\mathcal{K}$ functions.

Finally, by [11, Lemma 3.5], if (15) admits a LISS-Lyapunov function as in Definition 2, then (15) is LISS.

In what follows, we assume that an unforced periodic orbit that lies entirely on $\mathcal{Z}$ exists; by hybrid invariance such periodic orbits correspond to fixed points of (11) that have the form $(0, \xi^*)$ where $\xi^* = [(\theta^-)^* \ \zeta^*]^{\mathrm{T}}$. With the notation and definitions provided above, the main result can be stated as follows.

Theorem 1: Suppose $(0, \xi^*) \in \mathcal{S}$ where $\xi^* = [(\theta^-)^* \ \zeta^*]^{\mathrm{T}}$ is a 0-input equilibrium point of (11), and $\zeta^* \in \mathcal{Z} \cap \mathcal{S}$ is the corresponding 0-input equilibrium point of (14). Suppose there exists $\bar{F} > 0$ such that the forced reduced-order Poincaré map ρ in (14) is LISS. Then, there exists an $\epsilon^* > 0$ and a $\delta > 0$ such that for $(\eta, \xi) \in B_\delta(0, \xi^*) \cap \mathcal{S}$, the forced Poincaré map P^ϵ is LISS for all $F_{\mathrm{e}} \in \mathcal{F}$, and for all $\epsilon \in (0, \epsilon^*)$.

IV. Proof of Main Result

The proof of Theorem 1 is organized in a sequence of lemmas. We begin with a simple lemma showing that the reduced system (14) is locally ISS.

Lemma 1: Let $\zeta^* \in \mathcal{Z} \cap \mathcal{S}$ be a 0-input equilibrium point of (14). Assume that $F_{\mathrm{e}} \in \mathcal{F}$ and that $\bar{F} > 0$ is such that (13) is well defined. Then, the system (14) is LISS.

Proof: Consider $V_\zeta(\zeta) := (\zeta - \zeta^*)^2$. Then, by (13),

$$V_\zeta(\rho(\zeta, w_k)) - V_\zeta(\zeta) \leq -c_1|\zeta - \zeta^*|^2 + c_2|w_k| \ , \quad (18)$$

where $c_1 = \delta_{\mathrm{z}}^4(1 - \delta_{\mathrm{z}}^4) > 0$ (by [12, Section IV-A], $\delta_{\mathrm{z}} < 1$) and $c_2 = \left(1 + \frac{\delta_{\mathrm{z}}^4}{(1-\delta_{\mathrm{z}}^4)^2}\right) > 0$. By Definition 2, V_ζ is a LISS-Lyapunov function with $\alpha_1(r) = 0.5r^2$, $\alpha_2(r) = 2r^2$, $\alpha_3(r) = c_1 r^2$, $\alpha_4(r) = c_2 r$, and [11, Lemma 3.5] completes the proof. ∎

Remark 3: For subsequent use, note that $|w_k| \leq c_3 F_{\sup}$,

$$c_3 = \int_{\theta^+}^{\theta^-} \frac{\|\kappa_3(\xi)\|}{|\kappa_1(\xi)|} d\xi \ , \quad (19)$$

so that (18) can be re-written as

$$V_\zeta(\rho(\zeta, w_k)) - V_\zeta(\zeta) \leq -c_1|\zeta - \zeta^*|^2 + \alpha_\zeta(F_{\sup}) \ , \quad (20)$$

where $\alpha_\zeta(r) := c_2 c_3 r$ is a class-$\mathcal{K}$ function.

The structure of the proof of Theorem 1 follows that of the proof of [3, Theorem 2]. The appearance of the forcing term in (12), however, calls for modifications that are necessary to establish LISS of the forced system (12). The following lemma establishes the closeness of the ζ-component P_ζ^ϵ of the Poincaré map P^ϵ in (12) with its value ρ for $\epsilon = 0$.

Lemma 2: Let $(0, \xi^*)$ be a fixed point of the 0-input system (11). There exists a $\delta > 0$ such that for all $\epsilon > 0$, $(\eta, \xi) \in B_\delta(0, \xi^*) \cap \mathcal{S}$ and $F_{\mathrm{e}} \in \mathcal{F}$ with $\bar{F}$ sufficiently small as in Lemma 1 such that

$$|P_\zeta^\epsilon(\eta, \zeta, F_k) - \rho(\zeta, w_k)| \leq \lambda\|\eta\| \quad (21)$$

for some $\lambda > 0$.

A sketch of the proof of Lemma 2 is given in the Appendix.

With the aid of Lemma 2 we can now proceed with the proof of Theorem 1

Proof: [Theorem 1] Our objective is to construct a LISS-Lyapunov function V for the discrete-time nonlinear system (12). In a way analogous to the proof of [3, Theorem 2], the resulting LISS-Lyapunov function will have the form

$$V(\eta, \zeta) := V_\zeta(\zeta) + \sigma V_\eta(\eta) \quad (22)$$

where $\sigma > 0$ is a parameter determined by the procedure.

1) Construction of V_η: We begin by defining

$$V_\eta(\eta) := \eta^{\mathrm{T}} S \eta \ , \quad (23)$$

where S satisfies Lyapunov's equation $A^{\mathrm{T}}S + SA = -Q$ for a symmetric positive definite Q. Following arguments similar to [3, Section III] it can be shown that $\dot{V}_\eta \leq -\frac{\gamma}{\epsilon} V_\eta$ where $\gamma := \frac{\lambda_{\min}(Q)}{\lambda_{\max}(S)}$. By the comparison lemma [15, Lemma 3.4] $V_\eta(\eta(t)) \leq \mathrm{e}^{-\gamma t/\epsilon} V_\eta(\eta(0))$, and since

$$\lambda_{\min}(S)\|\eta\|^2 \leq V_\eta(\eta) \leq \lambda_{\max}(S)\|\eta\|^2 \ , \quad (24)$$

we have

$$\begin{aligned} V_\eta(P_\eta^\epsilon(\eta, \zeta, F_k)) &\leq \mathrm{e}^{-\gamma T_I^\epsilon(\eta,\xi,F_k)/\epsilon} \lambda_{\max}(S)\|\eta(0)\|^2 \\ &\leq \mathrm{e}^{-\gamma(T^*-\tau)/\epsilon} \lambda_{\max}(S) L_{\Delta_\eta}^2 \|\eta\|^2 \ , \end{aligned} \quad (25)$$

where the last inequality follows from (36) in the proof of Lemma 2 given in the Appendix and from the fact that due to hybrid invariance $\Delta_\eta(0, \xi) = 0$ so that $\|\Delta_\eta(\eta, \xi)\|^2 = \|\Delta_\eta(\eta, \xi) - \Delta_\eta(0, \xi)\|^2 \leq L_{\Delta_\eta}^2 \|\eta\|^2$ for $L_{\Delta_\eta} > 0$; see Remark 1. Then, by (25) and the first inequality of (24), the difference in the value of V_η between two consecutive steps satisfies the estimate

$$V_\eta(P_\eta^\epsilon(\eta, \zeta, F_k)) - V_\eta(\eta) \leq (c_\eta(\epsilon) - \lambda_{\min}(S))\|\eta\|^2 \ , \quad (26)$$

where $c_\eta(\epsilon) := \mathrm{e}^{-\gamma(T^*-\tau)/\epsilon} \lambda_{\max}(S) L_{\Delta_\eta}^2$. By continuity of c_η in ϵ, and since $\lim_{\epsilon \to 0^+} c_\eta(\epsilon) = 0$, there exists $\epsilon^* > 0$ so that $(c_\eta(\epsilon) - \lambda_{\min}(S)) < 0$ for all $\epsilon \in (0, \epsilon^*)$

2) Construction of V_ζ: We now turn our attention to V_ζ in (22). Motivated by the proof of Lemma 1, choose

$$V_\zeta(\zeta) = (\zeta - \zeta^*)^2 \ . \quad (27)$$

We can bound V_ζ by $0.5|\zeta - \zeta^*|^2 \leq V_\zeta(\zeta) \leq 2|\zeta - \zeta^*|^2$. The change in the value of V_ζ between successive steps is

$$\begin{aligned} V_\zeta(P_\zeta^\epsilon(\eta, \zeta, F_k)) - V_\zeta(\zeta) = \\ \left[V_\zeta(P_\zeta^\epsilon(\eta, \zeta, F_k)) - V_\zeta(\rho(\zeta, w_k))\right] \\ + \left[V_\zeta(\rho(\zeta, w_k)) - V_\zeta(\zeta)\right] \ . \end{aligned} \quad (28)$$

The term in the second square bracket in (28) satisfies

$$V_\zeta(\rho(\zeta, w_k)) - V_\zeta(\zeta) \leq -c_1|\zeta - \zeta^*|^2 + \alpha_\zeta(F_{\sup}) \quad (29)$$

from Remark 3. Regarding the first term in (28), defining $\Delta V_\zeta := V_\zeta(P_\zeta^\epsilon(\eta, \zeta, F_k)) - V_\zeta(\rho(\zeta, w_k))$ and using (27)

$$\begin{aligned} \Delta V_\zeta &= (P_\zeta^\epsilon(\eta, \zeta, F_k) - \zeta^*)^2 - (\rho(\zeta, w_k) - \zeta^*)^2 \\ &\leq \left(|P_\zeta^\epsilon(\eta, \zeta, F_k) - \zeta^*| + |\rho(\zeta, w_k) - \zeta^*|\right) \\ &\quad\ |P_\zeta^\epsilon(\eta, \zeta, F_k) - \rho(\zeta, w_k)| \ , \end{aligned} \quad (30)$$

where the last inequality is obtained through the triangle inequality. We have

$$\begin{aligned} |P_\zeta^\epsilon(\eta, \zeta, F_k) - \zeta^*| &= |P_\zeta^\epsilon(\eta, \zeta, F_k) - \rho(\zeta, w_k) + \rho(\zeta, w_k) - \zeta^*| \\ &\leq |P_\zeta^\epsilon(\eta, \zeta, F_k) - \rho(\zeta, w_k)| + |\rho(\zeta, w_k) - \zeta^*|. \end{aligned}$$

Substituting in (30) results in

$$\begin{aligned}\Delta V_\zeta \leq &\big(|P^\epsilon_\zeta(\eta,\zeta,F_k)-\rho(\zeta,w_k)|+2|\rho(\zeta,w_k)-\zeta^*|\big)\\ &|P^\epsilon_\zeta(\eta,\zeta,F_k)-\rho(\zeta,w_k)| \ .\end{aligned} \tag{31}$$

Since $\zeta^*=\rho(\zeta^*,0)=\delta_z^2\zeta^*-v$ by (13), we have

$$\begin{aligned}|\rho(\zeta,w_k)-\zeta^*| &\leq |\delta_z^2(\zeta-\zeta^*)|+|w_k|\\ &\leq \delta_z^2|\zeta-\zeta^*|+c_3F_{\sup} \ ,\end{aligned} \tag{32}$$

where c_3 is defined in Remark 3. Substituting (21) and (32) in (31), ΔV_ζ satisfies the following bound

$$\begin{aligned}\Delta V_\zeta &\leq \left(\lambda\|\eta\|+2\delta_z^2|\zeta-\zeta^*|+2c_3F_{\sup}\right)\lambda\|\eta\|\\ &\leq 2\lambda^2\|\eta\|^2+2\lambda\delta_z^2\|\eta\|\cdot|\zeta-\zeta^*|+c_3^2F_{\sup}^2 \ ,\end{aligned} \tag{33}$$

where $2\lambda c_3\|\eta\|F_{\sup}$ was written into only-state and only-force terms via $2ab\leq a^2+b^2$. Adding (29) and (33) gives

$$\begin{aligned}V_\zeta(P^\epsilon_\zeta(\eta,\zeta,F_k))-V_\zeta(\zeta) \quad &\leq c_4\|\eta\|^2+c_5\|\eta\|\cdot|\zeta-\zeta^*|\\ &\quad -c_1|\zeta-\zeta^*|^2+\alpha_4(F_{\sup}),\end{aligned} \tag{34}$$

where $c_4=2\lambda^2$, $c_5=2\lambda\delta_z^2$ are positive constants and α_4 is a class-$\mathcal{K}$ function defined as $\alpha_4(r):=\alpha_\zeta(r)+c_3^2r^2$, with $\alpha_\zeta(r)$ defined in Remark 3.

3) Construction of V: With V_η and V_ζ defined by (23) and (27), and using the estimates derived so far we now proceed with showing that V defined in (22) is a LISS-Lyapunov function for the system (12). First, note that

$$\begin{aligned}\min\{0.5,\sigma\lambda_{\min}(S)\}\|(\eta,\zeta-\zeta^*)\|^2 &\leq V(\eta,\zeta)\\ &\leq (2+\sigma\lambda_{\max}(S))\,\|(\eta,\zeta-\zeta^*)\|^2,\end{aligned}$$

satisfying (16) in Definition 2. Then, let $\Delta V := V(P^\epsilon(\eta,\zeta,F_k))-V(\eta,\zeta)$ and $\sigma>0$ a constant to be selected, and use the bounds (26) and (34), and (22) to get

$$\begin{aligned}\Delta V \quad &\leq \quad -c_1|\zeta-\zeta^*|^2+c_5\|\eta\|\cdot|\zeta-\zeta^*|\\ &\quad\quad -\left(\sigma(\lambda_{\min}(S)-c_\eta(\epsilon))-c_4\right)\|\eta\|^2+\alpha_4(F_{\sup})\\ &= \quad -\begin{bmatrix}|\zeta-\zeta^*| & \|\eta\|\end{bmatrix}\Lambda(\epsilon)\begin{bmatrix}|\zeta-\zeta^*|\\ \|\eta\|\end{bmatrix}+\alpha_4(F_{\sup}) \ ,\end{aligned}$$

where

$$\Lambda(\epsilon):=\begin{bmatrix}c_1 & -\frac{1}{2}c_5\\ -\frac{1}{2}c_5 & \sigma(\lambda_{\min}(S)-c_\eta(\epsilon))-c_4\end{bmatrix} \ .$$

On checking the leading principal minors of $\Lambda(\epsilon)$ we observe that $c_1>0$ and for $0<\epsilon<\epsilon^*$ we can choose $\bar\sigma$ as

$$\bar\sigma=\frac{1}{\lambda_{\min}(S)-c_\eta(\epsilon)}\left[\frac{c_5^2}{4c_1}+c_4\right] \ ,$$

such that for all $\sigma>\bar\sigma$, $\Lambda(\epsilon)$ is positive definite. Then,

$$\begin{aligned}\Delta V \quad &\leq \quad -\lambda_{\min}(\Lambda(\epsilon))\|\begin{bmatrix}|\zeta-\zeta^*| & \|\eta\|\end{bmatrix}^{\mathrm{T}}\|^2+\alpha_4(F_{\sup})\\ &= \quad -\lambda_{\min}(\Lambda(\epsilon))\|(\eta,\zeta-\zeta^*)\|^2+\alpha_4(F_{\sup}) \ .\end{aligned}$$

Choose $\alpha_1(r)=\min\{0.5,\sigma\lambda_{\min}(S)\}\,r^2$, $\alpha_2(r)=(2+\sigma\lambda_{\max}(S))\,r^2$, $\alpha_3(r)=\lambda_{\min}(\Lambda(\epsilon))r^2$, and $\alpha_4(r)=\alpha_\zeta(r)+c_3^2r^2$, thereby satisfying (16) and (17) in Definition 2. Hence, V is a LISS-Lyapunov function and [11, Lemma 3.5] implies that the system (12) is LISS. ■

V. Simulation Results

To demonstrate the behavior of the biped under piecewise constant and bounded forces, we present the evolution of the LISS-Lyapunov function V constructed in Theorem 1 over a number of steps; see Fig. 2. Two different forcing intensities are considered, and compared with the unforced response of the system. In the results shown in Fig. 2, the external force F_k applied over the k-th step acts in the horizontal direction, and its magnitude is obtained by sampling a uniform distribution so that $\|F_k\|\leq F_{\sup}$. Two values of $F_{\sup}$ are shown in Fig. 2; namely, $F_{\sup}=3$N and $F_{\sup}=5$N. In constructing the LISS-Lyapunov function (22) we used $\sigma=10$, $\epsilon=0.1$.

The unforced response of the system to perturbations away from its nominal orbit is shown by red crosses in Fig. 2. As expected, in the absence of the force, the Lyapunov function converges to zero asymptotically and stays there. In the presence of an external force with $F_{\sup}=3$N, the evolution of the values of the Lyapunov function is shown by grey triangles in Fig. 2. Initially, V decreases monotonically until it crosses the dashed-dot black line in Fig. 2. While the values of V remain below for all future steps, convergence to zero is not realized due to the persistent external force. Similarly when $F_{\sup}=5$N, the evolution of V shown by blue circles in Fig. 2 remains trapped below the dashed line in Fig. 2. Notice that the higher the value of $F_{\sup}$ is, the higher the threshold of the value V is (in Fig. 2, 33 for $F_{\sup}=3$N and 55 for $F_{\sup}=5$N). This is consistent with the expected behavior of an input-to-state stable system.

Finally, note that for $\epsilon>0.7$, the system becomes unstable, thus, affirming that the biped will keep taking steps in the presence of the external force only for sufficiently fast rates of convergence of the transversal dynamics as indicated by Theorem 1. It is worth noting that for $\epsilon>0.7$, the unforced response of the system was also unstable.

VI. Conclusion

This paper proves local input-to-state stability (LISS) for a dynamically walking biped under persistent external excitation in the form of a piecewise constant force which can change in a step-by-step fashion. First, a HZD controller is designed to generate locally exponentially stable unforced

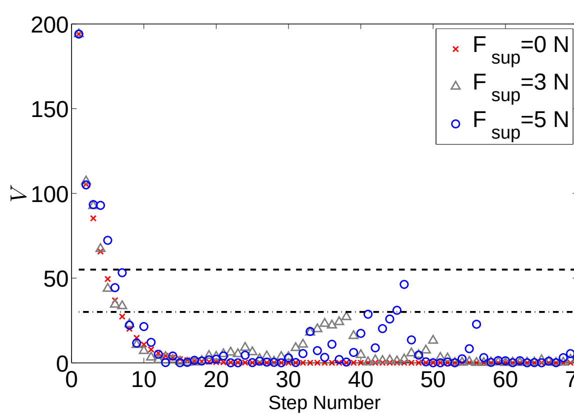

Fig. 2. Evolution of the LISS Lyapunov function V.

periodic gaits. Then, in a way analogous to [3], an ISS-Lyapunov function is constructed locally around the nominal orbit, establishing LISS for sufficiently fast convergence of the dynamics transversal to the zero dynamics manifold. Our motivation stems from a class of tasks that require the physical cooperation between a walking bipedal robot and an external agent (another robot or a human). In such tasks, the bipedal robot has to adapt its walking motion to external activity rather than trying to reject it. The results in this paper provide a first step towards a framework within which such tasks can be rigorously analyzed.

Appendix

Proof: [Lemma 2] As in the proof of [3, Lemma 1], let $\mu_1 \in \mathbb{R}^{\dim(\eta)}$ and $\mu_2 \in \mathbb{R}^{\dim(\xi)}$, and define

$$T_{\mathrm{B}}(\mu_1, \mu_2, \xi, F_k) := \inf \{t \geq 0 \mid p_{\mathrm{E}}^{\mathrm{v}}(\mu_1, \varphi_\xi(t, \Delta(0,\xi), F_k) + \mu_2) = 0\},$$

where $p_{\mathrm{E}}^{\mathrm{v}}$ is as in the definition (3) of $\mathcal{S}$, and φ_ξ is the continuous-time flow of the forced system on $\mathcal{Z}$ based on initial conditions $\Delta(0,\xi) \in \mathcal{Z}$. As in Remark 2, the implicit function theorem implies the existence of open sets $B_\delta(0,\xi^*)$ and $B_{\bar{F}}(0)$ over which T_{B} is well defined. Furthermore, by [14, Theorem 1.1], T_{B} is locally Lipschitz in μ_1, μ_2 and F_k. Noting that $T_I(0,\xi,F_k) = T_{\mathrm{B}}(0,0,\xi,F_k)$ we can write

$$|T_{\mathrm{B}}(\mu_1,\mu_2,\xi,F_k) - T_I(0,\xi,F_k)| \leq L_{\mathrm{B}}\left(\|\mu_1\| + \|\mu_2\|\right). \tag{35}$$

Let $\xi^\epsilon(t)$ and $\xi(t)$ denote the evolution of ξ for $\epsilon > 0$ and $\epsilon = 0$, respectively. Setting $\mu_1 = \eta(t)|_{t=T_I^\epsilon(\eta,\xi,F_k)}$ and $\mu_2 = \xi^\epsilon(t)|_{t=T_I^\epsilon(\eta,\xi,F_k)} - \xi(t)|_{t=T_I^\epsilon(\eta,\xi,F_k)}$ leads to $T_{\mathrm{B}}(\mu_1,\mu_2,\xi,F_k) = T_I^\epsilon(\eta,\xi,F_k)$ locally around the unforced fixed point $(0,\xi^*)$ and for $\|F_k\| < \bar{F}$; see [3, Lemma 1].

By Remark 2, T_I is continuous. Then, for $\tau > 0$ there exist $\delta > 0$ and $\bar{F} > 0$ (shrink if necessary) so that for $(\eta,\xi) \in B_\delta(0,\xi^*)$ and $F_k \in B_{\bar{F}}(0)$ we have $|T_I^\epsilon(\eta,\xi,F_k) - T_I^\epsilon(0,\xi^*,0)| \leq \tau$. Since $T_I^\epsilon(0,\xi^*,0) = T^*$, where T^* is the period of the unforced fixed point,

$$T^* - \tau \leq T_I^\epsilon(\eta,\xi,F_k) \leq T^* + \tau\ . \tag{36}$$

As mentioned in Remark 1, Δ is locally Lipschitz, so that

$$\|\Delta(\eta,\xi) - \Delta(0,\xi)\| \leq L_\Delta \|\eta\| \tag{37}$$

for some $L_\Delta > 0$. On the other hand, by (9) we have

$$\|\eta(t)\| \leq \beta_\ell \mathrm{e}^{-\frac{\lambda_\ell}{\epsilon}t}\|\eta(0)\|\ , \tag{38}$$

and by using the Gronwall-Bellman inequality in view of the fact that f_ξ and g_ξ are locally Lipschitz we obtain

$$\begin{aligned}\|\xi^\epsilon(t) - \xi(t)\| \leq{}& \|\xi^\epsilon(0) - \xi(0)\|\mathrm{e}^{Lt} \\ &+ \epsilon\frac{\beta_\ell L}{\lambda_\ell + \epsilon L}\left(\mathrm{e}^{Lt} - \mathrm{e}^{-\frac{\lambda_\ell}{\epsilon}t}\right)\|\eta(0)\|\end{aligned} \tag{39}$$

in which λ_ℓ, β_ℓ are positive constants, and $L := L_f + L_g\bar{F}$ where $L_f > 0$ and $L_g > 0$ are Lipschitz constants for f_ξ and g_ξ; see Remark 1. Noting that $0 < \mathrm{e}^{-\frac{\lambda_\ell}{\epsilon}t} \leq 1$, then using (36) in (38) and (39) followed by (37) we can find bounds

$$\|\mu_1\| \leq L_\Delta \beta_\ell \|\eta\| \text{ and } \|\mu_2\| \leq L_\Delta\left[1 + \beta_\ell\right]\mathrm{e}^{L(T^*+\tau)}\|\eta\|\ ,$$

which by (35) imply $|T_I^\epsilon(\eta,\xi,F_k) - T_I(0,\xi,F_k)| \leq L_\Delta L_{\mathrm{B}}\big[(1+\beta_\ell)\mathrm{e}^{L(T^*+\tau)} + \beta_\ell\big]\|\eta\|$. Letting $\lambda_1 = \max_{T^*-\tau\leq t\leq T^*+\tau}\|f_\xi(0,\xi) + g_\xi(0,\xi)F_k\|$, we have

$$\begin{aligned}\|\tilde{P}_\xi^\epsilon(\eta,\xi,F_k) - \tilde{P}_\xi(0,\xi,F_k)\| \\ \leq{}& \Big\|\xi^\epsilon(0) - \xi(0) \\ &+ \int_0^{T_I^\epsilon(\eta,\xi,F_k)}(f_\xi(\eta(s),\xi^\epsilon(s)) - f_\xi(0,\xi(s)))ds \\ &+ \int_0^{T_I^\epsilon(\eta,\xi,F_k)}(g_\xi(\eta(s),\xi^\epsilon(s)) - g_\xi(0,\xi(s)))F_k ds)\Big\| \\ &+ \left|\int_{T_I^\epsilon(\eta,\xi,F_k)}^{T_I(0,\xi,F_k)}\|f_\xi(0,\xi(s)) + g_\xi(0,\xi(s))F_k\|ds\right| \\ \leq{}& \|\mu_2\| + \lambda_1|T_I^\epsilon(\eta,\xi,F_k) - T_I(0,\xi,F_k)| \\ \leq{}& \lambda\|\eta\|\end{aligned}$$

where $\lambda = L_\Delta\left[(1+\lambda_1 L_{\mathrm{B}})(1+\beta_\ell)\mathrm{e}^{L(T^*+\tau)} + \lambda_1 L_{\mathrm{B}}\beta_\ell\right]$. The proof follows by noticing that for $(\eta,\xi) \in \mathcal{S}$ and in the neighborhood of the fixed point that we work, $\tilde{P}_\xi^\epsilon(\eta,\xi,F_k) = [\theta^-(\eta)\ \ P_\zeta^\epsilon(\eta,\zeta,F_k)]^{\mathrm{T}}$ so that $|P_\zeta^\epsilon(\eta,\zeta,F_k) - \rho(\zeta,w_k)| \leq \|\tilde{P}_\xi^\epsilon(\eta,\xi,F_k) - \tilde{P}_\xi(0,\xi,F_k)\| \leq \lambda\|\eta\|$. ■

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