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Stabilization and Output Tracking for Underactuated Mechanical Systems with Inequality State Constraints

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Abstract— **The paper presents a method to determine the feasibility of stabilization to an equilibrium manifold or exact output tracking for underactuated mechanical systems that are subject to inequality state constraints. Even for minimum phase systems, internal dynamics may evolve in an unacceptable way and has to be confined within certain limits. Such restrictions arise in deformable object manipulation tasks. It is shown that the problem of output tracking under inequality state constraints is equivalent to output tracking with bounded input. The paper provides sufficient conditions for exact output tracking and stabilization to an equilibrium manifold that guarantee that internal dynamics is bounded with adjustable bounds.**

I. Introduction

Underactuated systems are systems with fewer control inputs than degrees of freedom. This is a broad class of systems including mobile robots, gymnastic robots, underwater vehicles and surface vessels, VTOL aircraft and space robots. Control design for such systems is usually complicated due to non integrable first or second order nonholonomic constraints and non-minimum phase zero dynamics. Although underactuated systems control is still an open problem, several interesting results on stabilization and tracking have appeared in literature for specific classes of underactuated systems.

The stabilization problem for underactuated systems has received much attention lately. Results on stabilization have appeared for the cart-pendulum [1], the Acrobot and the gymnast robot [2], [3], planar underactuated manipulators [4], [5], [6], [7], [8], surface vessels [8], [9], [10]. In most cases the systems exhibit nonholonomic constraints and do not lend themselves to the application of smooth state stabilizing feedback.

Tracking for some classes of underactuated systems has recently received significant attention, particularly for the case of surface vessels [11]. Output tracking methodologies have been developed for surface vessels [12], for VTOL aircraft [13], [14] and for underwater vehicles [15]. Other problems related to underactuated systems include trajectory generation [16] and motion planning [17].

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Research has mainly focused on several classes of underactuated systems with specific application interest. For output tracking, in particular, no general methodology has been presented at least to the authors knowledge. State tracking methodologies could easily be adapted, yet the most general approach to tracking known to the authors [18] is limited to a the class for which the number of unactuated degrees of freedom is equal to the number of inputs.

Recently, deformable objects under manipulation have been studied as underactuated mechanical systems [19]. Object manipulation can benefit from the transfer of control engineering practices in underactuated systems. In object handling however, material strength limitations that can be expressed as inequality constraints impose severe restriction on the application of existing techniques. This motivated the development of a new methodology within this framework that enables one to determine whether a reference trajectory of the grasp points is feasible and if so, under what control law can it be realized. This problem is formulated as an output tracking problem for underactuated mechanical systems with inequality state constraints and solution is sought by using saturated linear feedback.

Considerable work has been done on the stabilization of linear systems and partially linear systems with input saturation [20], [21], [22], [23], [24]. In [22] it has been shown that multi-input linear systems with eigenvalues in the closed left-half complex plane can be semi-globally stabilized by saturated linear feedback. The semi-global restriction follows from an early result [25] that a linear system with a series of integrators of degree more than two cannot be globally stabilized using saturated linear feedback. Global stabilization has been achieved by using compositions of saturated controllers [21], [23], [24].

The rest of the paper is organized as follows: in section II the problem that motivated this work is formally stated within the framework of underactuated mechanical systems. Section III introduces the main result of the paper. The methodology is applied to a nontrivial underactuated mechanical system in section IV and simulation results are presented. Finally, section V summarizes the results of the present work.

II. Problem Statement

Consider the underactuated mechanical system:

$$M(z)\ddot{z} + C(z,\dot{z})\dot{z} + D\dot{z} + Kz + G(z) = F \quad (1)$$

where $z \in \Re^n$ is the configuration variables vector, $M(z)$ is the inertia matrix, $C(z,\dot{z})$ is the matrix of Coriolis and centrifugal terms, D is a positive definite matrix of the damping terms, K is the matrix of the elastic forces and G is the vector of gravitational forces. F is the vector of the input forces acting on the system.

Based on the equations that contain input terms, the configuration variables can be partitioned into two groups, $z = [z_1^T, z_2^T]^T$, the actuated part, $z_1 \in \Re^m$ and the unactuated part, $z_2 \in \Re^{n-m}$. Then (1) can be written:

$$m_{11}\ddot{z}_1 + m_{12}\ddot{z}_2 + c_1\dot{z} + d_1\dot{z} + k_1 z + g_1 = f \quad (2a)$$

$$m_{12}^T\ddot{z}_1 + m_{22}\ddot{z}_2 + c_2\dot{z} + d_2\dot{z} + k_2 z + g_2 = 0 \quad (2b)$$

The system above is supposed to be subject to the following inequality constraints:

$$\|z_2\|_\infty \leq \sigma \quad (3)$$

Let the output of the system be $y = [z_1^T, \dot{z}_1^T]^T$. The problem is stated as follows: Given the system (2a) -(2b) and a reference trajectory y_R, establish a sufficient condition under which the system (2a) -(2b) can track exactly the reference trajectory y_R, while the constraint $\|z_2\| \leq \sigma \in \Re_+$ is respected. If the condition is satisfied, then provide a control strategy to realize the reference trajectory.

For the stabilization to an equilibrium manifold case, the problem can be stated as follows: let $z_1 = z_{1d}$ be the desired equilibrium manifold for the system (2a) - (2b). Establish a condition under which stabilization to the equilibrium manifold can be achieved while $\|z_2\| \leq \sigma \in \Re_+$ at all time.

III. Approach to Solution

Due to the well known structural properties of the inertia matrix appearing in dynamic models of mechanical systems, system (2a) -(2b) can always be partially feedback linearized with respect to the actuated degrees of freedom z_1. This results in input-output feedback linearization of the original system and facilitates further analysis. Following the procedure in [26], (2b) is solved for $\ddot{z}_2$:

$$\ddot{z}_2 = -m_{22}^{-1}[m_{12}^T\ddot{z}_1 + c_2\dot{z} + d_2\dot{z} + k_2 z + g_2]$$

and plugging it into (2a):

$$\tilde{m}_{11}\ddot{z}_1 + \tilde{c}_1\dot{z} + \tilde{d}_2\dot{z} + \tilde{k}_2 z + \tilde{g}_2 = f$$

where

$$\tilde{m}_{11}(z) \triangleq m_{11}(z) - m_{12}(z)m_{22}^{-1}(z)m_{12}^T(z)$$
$$\tilde{c}_1(z) \triangleq c_1(z) - m_{12}(z)m_{22}^{-1}(z)c_2(z)$$
$$\tilde{d}_1(z) \triangleq d_1 - m_{12}(z)m_{22}^{-1}(z)d_2$$
$$\tilde{k}_1(z) \triangleq k_1 - m_{12}(z)m_{22}^{-1}(z)k_2$$
$$\tilde{g}_1(z) \triangleq g_1(z) - m_{12}(z)m_{22}^{-1}(z)g_2(z)$$

In a computed torque - like fashion, one can define

$$f = \tilde{c}_1\dot{z} + \tilde{d}_2\dot{z} + \tilde{k}_2 z + \tilde{g}_2 + \tilde{m}_{11}u$$

to obtain a system in the form:

$$\ddot{z}_1 = u \quad (4a)$$

$$\ddot{z}_2 = -m_{22}^{-1}m_{12}^T\ddot{z}_1 - m_{22}^{-1}(c_2\dot{z} + d_2\dot{z} + k_2 z + g_2) \quad (4b)$$

The analysis that follows is based on the lemma:

Lemma 1: The system (4b) when constrained to the manifold of $z_1 = z_{1d}$ is (at least) stable at the point z_{2e} defined by $k_2 z + g_2(z_{1d}, z_2) = 0$.

Proof: The proof is a consequence of the fact that the system (4b) derived after the partial feedback linearization described above is Lagrangian and therefore passive. On the manifold $z_1 = z_{1d}$, (4b) becomes:

$$m_{22}\ddot{z}_2 = -c_{22}\dot{z}_2 - d_{22}\dot{z}_2 - k_{22}z_2 - k_{21}z_{1d} - g_2 \quad (5)$$

Consider the positive definite storage function:

$$V = \frac{1}{2}\dot{z}^T M\dot{z} + \frac{1}{2}z^T K z + U$$

where U is the gravitational potential energy. Using (5) and the skew symmetry of $\dot{M} - 2C$, the time derivative of V when restricted to the manifold $z_1 = z_{1d}$ becomes

$$\dot{V} = -\dot{z}_2^T d_{22}\dot{z}_2$$

Furthermore, since

$$\dot{V} = 0 \Rightarrow \dot{z}_2 = 0 \overset{5}{\Rightarrow} k_2 z + g_2(z_{1d}, z_2) = 0$$

so that (5) output strictly passive relative to the output $\dot{z}_2$ and zero state observable [27]. This establishes the asymptotic stability of (4b) when confined on $z_1 = z_{1d}$. If $d_{22} \equiv 0$ then (5) is passive and therefore stable. ■

By using the transformation: $[\eta = [(z_2 - z_{2e})^T, \dot{z}_2^T]$ the system (4b) can be written as:

$$\dot{\eta} = w(\eta, v), \quad \text{where } v = [(z_1 - z_{1d})^T, \dot{z}_1^T, \ddot{z}_1^T]^T \quad (6)$$

Obviously, $w(0,0) = 0$. The input to state stability of (6) follows from lemma 1. Therefore:

$$\|\eta\|_\infty \leq \beta(\|\eta(0)\|_\infty, t - t_0) + \gamma(\sup\|v\|_\infty) \quad (7)$$

where β is of class $\mathcal{KL}$, and γ is of class $\mathcal{K}$. Finding an exact representation of for γ involves establishing an appropriate Lyapunov function. However, γ can be sufficiently approximated using a linearization of (6) at $\eta = 0, v = 0$:

$$\dot{\eta} = A_u\eta + B_u v \quad (8)$$

$$\text{where } A_u = \left.\frac{\partial w}{\partial \eta}\right|_{\eta=0,v=0}, \; B_u = \left.\frac{\partial w}{\partial v}\right|_{\eta=0,v=0}$$

Lemma 2: The matrix A_u appearing in (8) has eigenvalues in the closed left half plane.

Proof: It suffices to show that A_u cannot have eigenvalues with positive real part. This follows by contradiction since (6) is stable by lemma 1. ■

A. The Output Tracking Case

Contrary to stabilization, tracking requires the following assumption:

Assumption 1: The reference trajectory y_R can be followed by system (4a) with bounded control inputs $\|v\| \leq r$.

We are now in position to state our main results.

Proposition 1: Consider the system (4a) -(6), and let (8) be the linear approximation of (6) in a neighborhood N of $(z_1, \dot{z}_1, \eta) = (z_{1d}, 0, 0)$. Let μ be the minimum real number for which $(A_u + B_u H)$, with $\|H\eta\| \leq \mu, \forall\ \eta \in N$ is Hurwitz, and assume that assumption 1 holds. If the reference trajectory satisfies:

$$v_s - h - \mu - \sup\|\dot{z}_{1d}\| \geq s > 0$$

where h is given by (13) and v_s by (11), and in addition $s \leq r$, then there exists a linear state feedback law u that makes the reference trajectory locally exponentially stable for (4a) -(4b) while $\|z_2\|_\infty \leq \sigma$.

Proof: The case of A_u having eigenvalues with positive real part is excluded due to lemma 1.

Let A_u be Hurwitz. Then the Lyapunov equation provides a matrix $P(t)$ which serves as a Lyapunov function candidate. Then γ can be constructed as:

$$\gamma(\rho) \triangleq 4\sqrt{2(n-m)^3}\lambda_M(P)\sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}\rho$$

where $\lambda_M(P), \lambda_m(P)$ are upper and lower bounds for the eigenvalues of $P(t)$. For sufficiently small initial errors,

$$\|\eta\|_\infty \leq \gamma(\sup\|B_u v\|_\infty) \tag{9}$$

Given $\eta = [(z_2 - z_{2e})^T, \dot{z}_2^T]^T$,

$$\|\eta\|_\infty + \sup\|z_{2e}\|_\infty \geq \|z_2\|_\infty$$

Therefore,

$$\|\eta\|_\infty + \sup\|z_{2e}\|_\infty \leq \sigma \Rightarrow \|z_2\|_\infty \leq \sigma$$

which means that

$$\|\eta\|_\infty \leq \sigma - \sup\|z_{2e}\|_\infty \tag{10}$$

From the right hand sides of (9) and (10), it follows that a sufficient condition for (3) is

$$\begin{aligned}&\sup\|v\|_\infty \leq v_s(\sigma, y_R)\\ &\triangleq \frac{\sigma - \sup\|z_{2e}\|_\infty}{8(n-m)^2\lambda_M(P)\lambda_M(B_u)}\sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}} > 0\end{aligned} \tag{11}$$

where $\lambda_M(B_u)$ is an upper bound for the eigenvalues of B_u. Given $v = [(z_1 - z_{1d})^T, \dot{z}_1^T, u^T]^T$,

$$\|v\| \leq \|[e_y, \dot{e}_y, 0]\| + \|\dot{z}_{1d}\| + \|u\|$$

with $e_y = z_1 - z_{1d}$ and $\dot{e}_y = \dot{z}_1 - \dot{z}_{1d}$. Since e_y and $\dot{e}_y$ decay exponentially,

$$\sup\|v\|_\infty \leq \max\{\|e_y(0)\|_\infty, \|\dot{e}_y(0)\|_\infty\} + \sup\|\dot{z}_{1d}\| + \sup\|u\|$$

Thus for $\|z_2\|_\infty \leq \sigma$, it is sufficient

$$v_s - h - \sup\|\dot{z}_{1d}\| \geq \sup\|u\| = s > 0 \tag{12}$$

where

$$h = \max\{\|e_y(0)\|_\infty, \|\dot{e}_y(0)\|_\infty\} \tag{13}$$

Relation (12) is a sufficient condition for the feasibility of the reference trajectory. If it is satisfied for some positive $s \leq r$, then a linear control law $u = H[e_y^T, \dot{e}_y^T]^T$ can be constructed with $\|u\|_\infty \leq s$ [22] so that the system locally converges exponentially to the reference trajectory.

Consider now the case where A_u is not Hurwitz and assume that the pair (A_u, B_u) is controllable. Then a saturated linear control law $v = H\eta$ with $\|H\eta\| \leq \mu$ can be constructed [22] so that $(A_u + B_u H)$ is Hurwitz. Working as before we can find a matrix $P(t)$ for which the function $V(t, \eta) = \eta^T P(t)\eta$ is Lyapunov. The virtual control input v dictates trajectory errors e_r, $\dot{e}_r$ and a system input u_r so that (6) is locally exponentially stable:

$$H\eta = \begin{bmatrix} e_r^T & (\dot{e}_r + \dot{z}_{1d})^T & u_r^T \end{bmatrix}^T$$

If the tracking controller u and the corresponding errors do not perturb this system in a way that (3) is violated, then the output can be tracked exactly. If follows that in order for that to hold,

$$v_s - h - \mu - \sup\|\dot{z}_{1d}\| \geq \sup\|u\| = s > 0 \tag{14}$$

Then u can be designed in such a way that $\|u\|_\infty \leq s \leq r$ [22] so that (4a) -(4b) can locally exponentially track the reference trajectory and simultaneously satisfy (3). ■

Condition (14) agrees with engineering intuition: it is obvious that if one keeps velocities and accelerations *sufficiently* small, the unactuated degrees of freedom will remain bounded. The merit of (14) is mainly that one can immediately identify the class of admissible trajectories and control laws, without resorting to simulations or even worse, destructive experimentation.

In brief, proposition 1 states that if the reference trajectory is stabilizable using bounded input and this bound satisfies (14) then the trajectory can be tracked with simultaneous satisfaction of constraint (3).

B. The Stabilization Case

When referring to stabilization, we refer to stabilization to an equilibrium manifold $z_1 = z_{1d}$. For this case, assumption 1 is not needed. In fact, proposition 1 can be restated for the stabilization case by noting that $\dot{z}_{1d} = 0$ as follows:

Proposition 2: Consider the system (4a) -(6), and let (8) be the linear approximation of (6) in a neighborhood N of $(z_1, \dot{z}_1, \eta) = (z_{1d}, 0, 0)$. Let μ be the minimum real number for which $(A_u + B_u H)$, with $\|H\eta\| \leq \mu, \forall\ \eta \in N$ is Hurwitz. If μ satisfies

$$v_s - h - \mu \geq s > 0$$

where h is given by (13) and v_s by (11), then there exists a linear state feedback law u that makes the equilibrium manifold $z_1 = z_{1d}$ semi globally exponentially stable for (4a) -(4b) while $\|z_2\|_\infty \leq \sigma$.

Proof: The proof is almost identical to that of 1. By substituting $\dot{z}_{1d} = 0$ (14) becomes:

$$v_s - h - \mu \geq \sup\|u\| = s > 0$$

Then one can find a sufficiently small ε for which [22]

$$v = H(\varepsilon)[(z_{1d} - z_1)^T, \dot{z}_1^T]^T \leq s$$

in the region $\{z_1 | z_1 < h\}$. It follows that z_{1d} is a semi global exponentially stable equilibrium of (4a) ■

Note that for stabilization, local stabilization is relaxed to semi global. Unlike the tracking case, the equilibrium manifold is always stabilizable when $v_s - \mu \geq s$.

C. The Bounded Controller

The system (2a) - (2b) does not necessarily falls in the class of feedforward systems discussed in [20], [21] or the triangular systems in [24]. Therefore, these techniques cannot be readily applied except in cases where the aforementioned system enjoys such a structure. In the general case, and given the I/O feedback linearization of (2a) - (2b), the approach of [22] seems more appropriate. Moreover, the objective here is slightly different in that output tracking instead of state stabilization is sought and that the state is constrained to lay within a certain region during tracking.

In order for the method in [22] to be applied, the original open loop system must be at least marginally stable. By lemma 1 this holds for system (8), and therefore it is always possible to apply the results of [22] to (8).

The problem of stabilizing a linear system with eigenvalues in the closed left hand complex plane reduces to relocation of the eigenvalues located on the imaginary axis [22]. Broadly speaking, the control effort is proportional to this displacement. The new position of these eigenvalues is parametrized by ε so that their new position can be described by the set:

$$\Lambda(\varepsilon) = \{\varepsilon\bar{\lambda}_k + \omega_k \mathrm{i}, \varepsilon\bar{\lambda}_k\} \tag{15}$$

where $\bar{\lambda}_k < 0$, with $\varepsilon = 0$ giving the location of the eigenvalues of the open loop system. The rest of the procedure involves calculating the characteristic polynomials associated with the set (15) and the closed loop matrix $A + BH$ where $H = \varepsilon\bar{H}$ and forming equations with the respective coefficients. The resulting system of algebraic equations is usually undetermined. A procedure to obtain a solution for this system is described in [22].

IV. Application

The methodology introduced is applied to the problem of output tracking for the system depicted in Figure 1. It is a system composed of a cart moving horizontally on which a chain of two unactuated links is mounted. The objective is for the cart to follow a desired trajectory while the link angles are bounded within certain limits. It is worth mentioning that the technique in [18] is not applicable here because $m < n - m$.

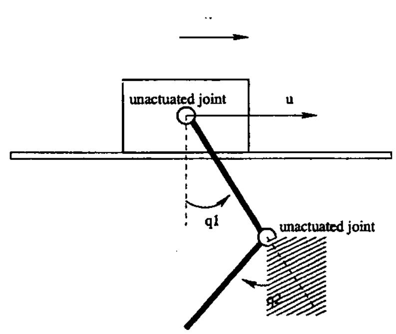

Fig. 1. A cart with two unactuated links

Using unity values for all dynamic parameters, the system equations in form (4a) - (4b) are as follows:

$$\begin{aligned}
\ddot{x} =& v \\
\ddot{q}_1 =& \frac{1}{r}\left[2\cos(q_1 + 2q_2) - 28\cos q_1 - 8\cos(q_1 + q2)v\right. \\
&+ 2\sin(q_1 + 2q_2) + 2\sin q_2 \left(\dot{q}_1^2(1 + 2\cos q_2)\right. \\
&\left.\left.+10\dot{q}_1\dot{q}_2 + 5\dot{q}_2^2\right) - 28\sin q_1 - 8\sin(q_1 + q_2)\right], \\
\ddot{q}_2 =& \frac{1}{r}\left[(12\cos(q_1 + q_2) + 2\cos(q_1 + 2q_2)\right. \\
&-4\cos q_1 - 6\cos(q_1 - q_2))\, v \\
&+ 4\sin q_1 + 6\sin(q_1 - q_2) - 12\sin(q_1 + q_2) \\
&- 2\sin(q_1 + 2q_2) - 4\dot{q}_1^2\left(5\sin(q_2) + \sin(2q_2)\right) \\
&\left.-4\dot{q}_1\dot{q}_2\left(\sin q_2 + \sin(2q_2)\right) - 2\dot{q}_2^2\left(\sin q_2 + \sin(2q_2)\right)\right]
\end{aligned}$$

where

$$r \triangleq 47 + 16\cos q_2 - 2\cos(2q_2)$$

Let $\eta_1 = q_1, \eta_2 = q_2, \eta_3 = \dot{q}_1, \eta_4 = \dot{q}_2$. The linearization of the unactuated subsystem at $q_1 = 0, q_2 = 0, v = 0$ gives:

$$\dot{\eta} = A\eta + Bv$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{34}{61} & -\frac{4}{61} & 0 & 0 \\ -\frac{4}{61} & -\frac{22}{61} & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ -\frac{34}{61} \\ -\frac{4}{61} \end{bmatrix}$$

The pair (A, B) is controllable and the open loop eigenvalues are $\lambda_{1,2}^o = \pm 0.7598i, \lambda_{3,4}^o = \pm 0.5838i$.

We assign the closed loop eigenvalues at $\lambda_{1,2}^c = -\varepsilon \pm 0.7598i, \lambda_{3,4}^c = -\varepsilon \pm 0.5838i$, where $\varepsilon > 0$ a small constant. The gain matrix $H(\varepsilon)$ that achieves that with $\|H(\varepsilon)\| \leq c\|\varepsilon\|$ is calculated in a straightforward manner from a square system of algebraic equations and given as:

$$\begin{aligned}
H = [&\frac{\varepsilon}{12}(56\varepsilon + 61\varepsilon^3) \quad \frac{\varepsilon}{24}(1244\varepsilon - 1073\varepsilon^3) \\
&\frac{\varepsilon}{3}(28 + 61\varepsilon^2) - \frac{\varepsilon}{6}(110 + 1037\varepsilon^2)]
\end{aligned}$$

With $v = H(\varepsilon)\eta$ the linearized unactuated system becomes exponentially stable.

Suppose that q_1, q_2 have to be confined to the interval $(-0.1, 0.1)$, so $\sigma = 0.1$. Then $\|v\| \leq c\|\varepsilon\|\sigma \triangleq \mu(\varepsilon)$. The solution of the Lyapunov equation provides the matrix P from which $v_s(\sigma, \varepsilon)$ can be calculated:

$$v_s = \frac{\sigma}{2\lambda_M(P)\lambda_M(B_u)}\sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}}$$

where the fact that $\sup\|z_{2e}\| = 0$ has been utilized and the $\|\cdot\|_2$ was used. Both v_s and μ are linear in σ so the sign of their difference is only depended on ε. It turns out (mainly due to the fact that A has all eigenvalues on the imaginary axis) that $v_s(\varepsilon) - \mu(\varepsilon) < 0$, no matter how small ε is chosen. Therefore, the sufficient condition for trajectory tracking (14) is not satisfied.

If however the system had some sort of damping (e.g. friction) at the joints then the method could exploit that to allow tracking. Assume that the system has damping terms of the form $d\dot{q}$ so that linearization of the internal dynamics results to:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{34}{61} & -\frac{4}{61} & -0.01 & 0 \\ -\frac{4}{61} & -\frac{22}{61} & 0 & -0.01 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -\frac{34}{61} \\ -\frac{4}{61} \end{bmatrix}$$

Then A is Hurwitz and the Lyapunov equation can be solved to yield P from which v_s is calculated. Specifically for this system, condition (14) can be relaxed by noting that $\dot{x}_d$ and x do not influence the linearized unactuated subsystem. Therefore, (14) can be replaced by $v_s > \sup\|v\|$. Then, choosing a trajectory that satisfies 1 and using the linear controller:

$$v = -\sqrt{2}\varepsilon(\dot{x}_d - \dot{x}) - \varepsilon^2(x_d - x)$$

where ε is such that $\|v\| < v_s$ for some bounded h, we can make the trajectory locally exponentially stable. This case is depicted in Figure 2. It is evident from Figure 2

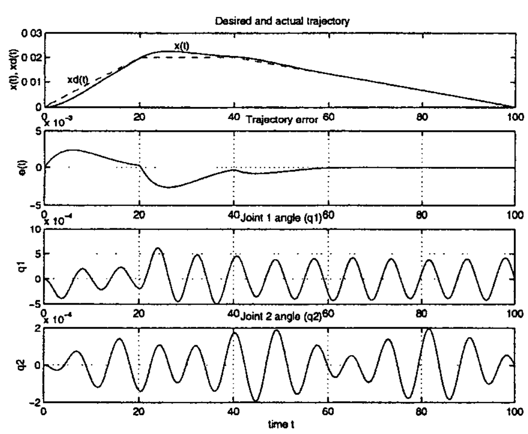

Fig. 2. Tracking with $\|q_i\| < 0.1$

that q_1 and q_2 remain bounded, well below the limit σ. This is due to the conservative nature of condition (14), which provides additional robustness against unmodeled dynamics and external disturbances.

The evolution of the control input v used to achieve tracking is also of interest. This is given in Figure 3. Figure

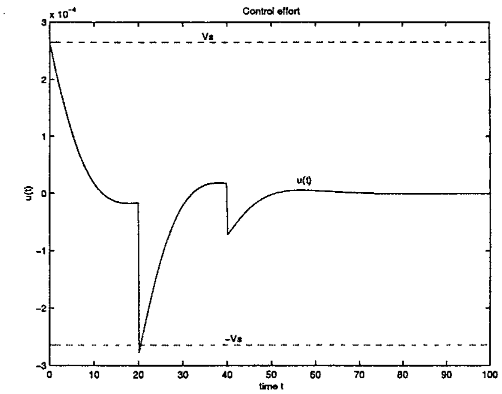

Fig. 3. Control input for $\sigma = 0.1$

3 shows that the control input v is bounded by the specification limit v_s set by $\|v\| < v_s$.

Then the bound σ is reduced by half. The reduction of σ affects the maximum admissible value for the tracking controller and therefore degrades performance in view of more restricting specifications for the variation of q_i. These effects are depicted in Figure 4. Figure 5 shows the evolution of the control input with time.

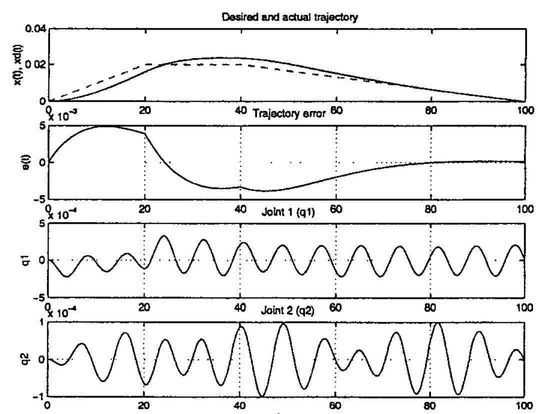

Fig. 4. Tracking with $\|q_i\| < 0.05$

V. Conclusion

This paper presents a procedure to determine the feasibility of exact output tracking and stabilization to an equilibrium manifold for underactuated mechanical systems which are subject to inequality state constraints. The systems considered in this work do not have to be minimum

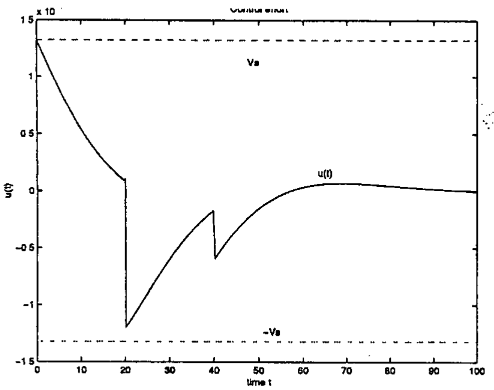

Fig. 5. Control input for $\sigma < 0.05$

phase nor have a special feedforward or triangular structure. It is shown that the output tracking problem under inequality state constraints is reduced to that of finding a sufficiently bounded tracking controller. In this perspective, a subset (since the condition provided is only sufficient) of the class of feasible trajectories for the system is characterized. Moreover, a methodology to design bounded feedback semi global controllers is indicated. For the case of stabilization to an equilibrium manifold, the satisfaction of the feasibility condition is on its own sufficient for stabilization. This feasibility condition provided is conservative enough to handle unmodeled dynamics and external disturbances.

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