Synchronous Rendezvous for Periodically Orbiting Vehicles with Very-Low-Range Interactions

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***Abstract*— This paper first offers conditions under which vehicles that are orbiting along adjacent paths will rendezvous in a small neighborhood intersecting with their paths, and then presents a controller that regulates their motion characteristics over a small time window in order to robustify their synchronous rendezvous from the first occurrence of that rendezvous, forward. The assumption is that the orbiting vehicles can only interact with each other when they are in close proximity, and they are moving independently otherwise. The analysis treats a pair of agents first, and then generalizes the approach to a chain of vehicles. Simulation results are presented to corroborate the theoretical analysis.**

I. Introduction

Synchronous rendezvous refers to the problem of spatially and temporally syncrhonizing a group of agents so that they meet with each other at prespecified locations [1]. In the particular context [1] where the term was introduced, the agents could only communicate and interact with each other when they were "on (rendezvous) point," considered to have very-low-range communication capabilities. Such a problem is encountered in cases where minimally actuated small drifters are deployed over a large stretch of water in which whirlpool-like geophysical flows are present, and one needs to get as many of them together at the same location to harvest via low-range radio the data they have collected, and allow those same drifters to clear their limited data-storage components and continue their data collection task. Beyond this application scenario that motivates this particular work, instances of the synchronous rendezvous problem can also be identified in other environmental sampling cases [2], in mine countermeasures involving Autonomous Underwater Vehicle (AUV) [3], in coordination of Unmanned Aerial Vehicle (UAV) systems [4], and satellite formation control [5].

While a lot of work has been devoted on consensus-type vehicle interactions, less is known on what is possible when the interaction range between the agents is very small relative to the scale of their deployment space [2], [6]. For example, buoys may be wirelessly networked over a few kilometers, and high-bandwidth underwater optical communication is only feasible within only a few meters. When those vehicles operate beyond communication range most of the time, only sporadic, brief, and intermittent interaction is possible.

This paper formulates a problem where agents, modeled as oscillators, are moving along paths that intersect with a very small spatial neighborhood. These oscillators can only interact with each other if they are collocated in that neighborhood and they do not have the option of stopping and waiting there; their motion is transient —because, for instance, the current will carry them away— and they can only regulate their speed of transition. The objective is to ascertain first if their original assignment of frequencies and phases will allow them to rendezvous sometime in the future, and if so, to design controllers that could act on them while they are in proximity with each other and coordinate their motion in order to (i) start meeting more regularly from that point on, and (ii) maximize the time they can interact with each other during each subsequent rendezvous.

This type of synchronous rendezvous problem bears similarities with rendezvous, and oscillator synchronization problems, respectively. In a typical rendezvous problem, agents need to arrive at the same place simultaneously, or at least within some common time interval [1], [7], [8]. The focus is on the agents converging the same position at the same time. In oscillator synchronization problems the objective is different: to synchronize certain parameters of these oscillatory motions —for example, frequency and/or phase difference— and keep them so over time [9], [10].

Solutions to both rendezvous and oscillator synchronization problems hinge on *connectivity*. Connectivity facilitates information sharing, which enables interaction between the agents, which in turn couples them dynamically via their control laws. In a typical rendezvous control law [11], this shared information takes the form of relative position. Often, information sharing is contingent on physical proximity [11], [12], yet still, agents find themselves communicating with neighbors more often than not. Depending on the strength of the dynamic coupling induced through cooperative control action, the time interval of information sharing and interaction, may vary. In oscillator synchronization instances [10], for example, this coupling strength is captured by a particular parameter, and the asymptotic behavior of the ensemble depends critically on the value of this parameter [10], [13], [14]. In a synchronous rendezvous instance [1], there may not be constraints on coupling strength necessarily, but usually there are stringent constraints on interaction time.

Classical rendezvous and synchronization setups differ from synchronous rendezvous problems in several ways. One is reflected in the steady-state: in rendezvous or synchronization problems, states or differences between agent states, asymptotically converge to constants. In synchronous rendezvous the steady state is some type of limit cycle. Another difference relates to an underlying assumption in both rendezvous or oscillator synchronization work, namely

This work was partially supported by ONR under #

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that agents are connected to some group-mate "most of the time;" the latter being captured mathematically in different forms [10], [11], [15]–[17]. This is not true in synchronous rendezvous [1]. Even though infrequent connectivity can still enable asymptotic convergence of the relevant parameters using existing techniques, the outcome could be unacceptable performance-wise, due to the long settling times forced by brief, intermittent agent interactions. Short "waiting periods" over rendezvous locations have proven sufficient for some cases of group synchronization [1]. The inability to wait at the rendezvous location for your partner(s) to arrive, however, complicates the problem.

Synchronous rendezvous where very-low-range interacting agents cannot wait at rendezvous locations, and the coordinating control action is not instantaneous, appears to be partially reducible to *integer programming*. Specifically, the answer to the question whether and when two agents will rendezvous is the solution to an integer programming problem. This analysis informs on the formulation of sufficient conditions on the agents initial phases and frequencies in order for this rendezvous to ever occur. Once one knows that rendezvous will happen, agents may want to utilize control action to make their subsequent rendezvous longer.

The paper models the agents as harmonic oscillators and the analysis takes place in a single dimension. To derive sufficient conditions for rendezvous, is a technique called *hyperplane decomposition* on lattices is used. This technique is part of a much more general methodology that is sometimes referred to as *algorithmic geometry of numbers* [18] from where integer programming borrows from. The paper derives conditions for synchronous rendezvous within this mathematical framework, and then proceeds with the design of motion controllers that regulate the agents' speed locally, in the small region where they can interact with each other, so that in subsequent encounters they have as much more interaction time together as possible. The last part of the technical discussion in this paper treats an extension to chains of oscillators on a line, suggesting a synchronization policy over different rendezvous points on this line.

II. Technical Preliminaries

A. Hyperplane Decompositions and Lattice Width

Consider a d-dimensional lattice $\mathcal{L} \subseteq \mathbb{Z}^d$ embedded in the $\mathbb{R}^d$ plane (see Fig. 1 for a 2-dimensional example). Each lattice point is labeled by an integer pair of coordinates. The *feasibility* question for an Integer Linear Programming (ILP) problem reduces to finding whether a *convex body* K (a full-dimensional convex compact set) includes lattice points, i.e., if $K \cap \mathcal{L} \neq \varnothing$. The answer to this question ultimately relates to how "thin" K might be, which requires a notion of measure for width.

Projective geometry [19] suggests the definition of *hyperplanes*, which in the planar case of Fig. 1 take the form of (diagonal) lines passing through the lattice points. Although there can be several hyperplanes that can be defined in this way, the ones of interest here are the ones defined by means of *primitive* vectors in $v \in \mathbb{Z}^d$. Those are (nonzero) vectors

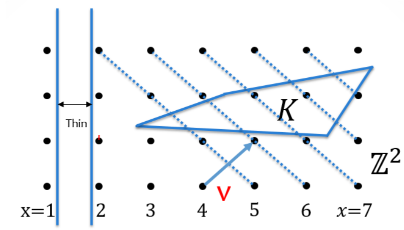


Fig. 1: A convex body on a hyperplane-decomposed $\mathbb{Z}^2$ lattice.

with components which are relatively prime. (If this is not the case for a vector v, the decomposition that it induces is not the coarsest possible [20].) With such a primitive at hand, $\mathbb{Z}^d$ can be characterized in relation to $\mathbb{R}^d$ [20] as $\mathbb{Z}^d \subseteq \bigcup_{z\in\mathbb{Z}} \{x \in \mathbb{R}^n \mid v^\intercal x = z\}$.

Definition 1 (Width [21]): . Let $K \subset \mathbb{R}^d$. The *width* of K along primitive $v \neq 0$ in $\mathbb{R}^d$ is defined as

$$\text{width}_K(v) = \max\{v^\intercal x \mid x \in K\} - \min\{v^\intercal x \mid x \in K\}$$

The *dual lattice* [20] of $\mathcal{L}$, denoted $\mathcal{L}^*$ is defined as $\mathcal{L}^* = \{y \in \mathbb{R}^d \mid y^\intercal v \in \mathbb{Z},\ \forall\, v \in \mathcal{L}\}$.

Definition 2 (Lattice width [21]): The *lattice width* of K is the minimal value of its width among all directions of the dual lattice $\mathcal{L}^*$, namely,

$$\text{width}(K, \mathcal{L}^*) = \min\{\text{width}_K(y) \mid y \in \mathcal{L}^* \setminus \{0\}\} \tag{1}$$

For a set $K \subset \mathbb{R}^n$, K^* represents its *polar*, i.e., $K^* = \{y \in \mathbb{R}^n \mid y^\intercal x \leq 1, \quad \forall\, x \in K\}$. In the same context, the notation $K - K$ is understood in a Minkovski sense, where $K - K = \{x - y \mid x \in K \ni y\}$. The following theorem is key to developing existence conditions for the solutions of ILP problems.

Theorem 1 (Khinchine's flatness theorem [22]): For any convex body $K \in \mathbb{R}^n$, either

$$\mu(K, \mathbb{Z}^n) \triangleq \inf\{s \geq 0 : \mathbb{Z}^n + sK = \mathbb{R}^n\} \leq 1 \quad \text{or}$$
$$\lambda_1\big((K-K)^*, \mathbb{Z}^n\big) \triangleq \inf_{v\in\mathcal{L}^*\setminus\{0\}} \text{width}_K(v) \leq f(n)$$

In the above, μ and $f(n)$ are the *covering radius* and the *width* function of K, respectively. The intuition behind the flatness theorem is that if the covering radius of the convex body is small, then it is bound to contain integer points, whereas if its width is sufficiently small (it is "fairly flat"), it can fit inside the lattice without intersecting with any of its points.

III. Problem Statement

A. Basic formulation

Consider N one-dimensional harmonic oscillators with identical amplitude A,

$$p_n = o_n + A\sin(\omega_n t + \phi_n)\ , \quad n = 1, \ldots, N$$

where $p_n \in \mathbb{R}$ denotes the position of oscillator n before rendezvous, $\omega_n > 0$ its frequency, ϕ_n its initial phase, and o_n its mid-point of oscillation satisfying

$$o_{n+1} = o_n + 2A \tag{2}$$

The rendezvous regions are small neighborhood of radius B around the point $o_{n,n+1} \triangleq o_n + A = o_{n+1} - A$, assuming in general that $B \ll A$. After some relabeling and variable transformations $x_{n\,1} := p_n - o_n$, $x_{n\,2} := \dot{p}_n/\omega_n$, and with the introduction of an agent control input u_n, the dynamics of these oscillators are expressed in state-space form, for $n \in \{1, \ldots, N\}$, as follows

$$\begin{bmatrix} \dot{x}_{1\,n} \\ \dot{x}_{2\,n} \end{bmatrix} = \begin{bmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{bmatrix} \begin{bmatrix} x_{1\,n} \\ x_{2\,n} \end{bmatrix} + \begin{bmatrix} 0 \\ u_n \end{bmatrix} \tag{3}$$
$$x_{1\,n}(0) = A \sin \phi_n \;, \qquad x_{2\,n}(0) = A \cos \phi_n$$

Agents can exchange information with each other —their positions and velocities, specifically— only when they are both within a rendezvous region. It is assumed that the agents can force a *single* instantaneous reset in their frequencies ω_n as part of their control strategy; besides that one permissible switch, control action is implemented through $u_n(t)$. Only when $p_n \in [o_n - A, o_n - A + B] \cup [o_n + A - B, o_n + A]$, can the reset in ω_n occur and $u_n(t) \neq 0$.

A rendezvous occurs when $p_n \in [o_n + A - B, o_n + A + B] \ni p_{n+1}$ for some $n \in \{1, \ldots, N\}$. No (prior) common information is assumed for the agent collection. The objective is, given a collection of parameters (A, ω_n, ϕ_n) for $n \in \{1, \ldots, N\}$, and under the communication constraints mentioned above, to (i) determine which pair $(n, n+1)$ of oscillators, if any, will rendezvous first, and if it does, (ii) to lock them into synchronous periodic rendezvous from this point on, maximizing the time they spend in rendezvous.

Compared to the original formulation of the technical problem for synchronous rendezvous [1], the above is different in a few aspects. The first relates to the agent dynamics, which are periodic in both cases but here they are actually harmonic. Another difference is that here rendezvous occurs not at discrete points (graph nodes on a bipartite graph), but rather at small but contiguous spatial regions which are treated uniformly. The most critical difference, however, between the two formulations may be the fact that here agents cannot remain stationary within rendezvous regions.

B. Possible Extensions

Given a solution to the problem stated in Section III-A, a natural follow-up question is whether this solution can be applied recursively to synchronize multiple pairs of agents, back-to-back on the same line.

IV. Technical Approach

A. Conditions for rendezvous

Focusing on a pair of agents n and $n+1$, let S_n^+ be the collection of disjoint time intervals in which agent n is in its upper rendezvous region, that is $S_n^+ \triangleq \bigcup\{[t_n^-, t_n^+] : p_n(t) > o_n + A - B, \; \forall\, t \in [t_n^-, t_n^+]\}$, and similarly, $S_{n+1}^- \triangleq \bigcup\{[t_{n+1}^-, t_{n+1}^+] : p_{n+1}(t) < o_n + A + B, \; \forall\, t \in [t_{n+1}^-, t_{n+1}^+]\}$. The pair will be in rendezvous at time instants $t_{n,n+1} \in S_n^+ \cap S_{n+1}^-$. These time instants are solutions to the system of inequalities

$$\begin{aligned} \sin(\omega_n\, t_{n,n+1} + \phi_n) &\geq 1 - \tfrac{B}{A} \\ \sin(\omega_{n+1}\, t_{n,n+1} + \phi_{n+1}) &\leq \tfrac{B}{A} - 1 \end{aligned} \tag{4}$$

Denote $\theta^+ \triangleq \arcsin \frac{A-B}{A}$ and $\theta^- \triangleq \arcsin \frac{B-A}{A}$, both of which range in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Combining with (4), $S_n^+ \cap S_{n+1}^-$ is refined in terms number of individual agent oscillation periods, namely k_n and k_{n+1}, respectively:

$$\bigcup_{k_n \in \mathbb{Z}} \left[\tfrac{2k_n\pi + \theta^+ - \phi_n}{\omega_n},\; \tfrac{2(k_n+1)\pi - \theta^+ - \phi_n}{\omega_n} \right]$$
$$\triangleq T_n^+ \ni t_{n,n+1} \in T_{n+1}^- \triangleq$$
$$\bigcup_{k_{n+1} \in \mathbb{Z}} \left[\tfrac{2(k_{n+1}-1)\pi - \theta^- - \phi_{n+1}}{\omega_{n+1}}, \tfrac{2k_{n+1}\pi + \theta^- - \phi_{n+1}}{\omega_{n+1}} \right]$$

For such an instant $t_{n,n+1}$ to exist, there must be $T_n^+ \cap T_{n+1}^- \neq \varnothing$. Every component of such a nonempty intersection corresponds to an integer pair $(k_n, k_{n+1}) \in \mathbb{Z}^2$. Setting

$$\overline{g}_{n,n+1} \triangleq \tfrac{\phi_n - \theta^+}{\omega_n} + \tfrac{\theta^- - \phi_{n+1}}{\omega_{n+1}} \tag{5a}$$
$$\underline{g}_{n,n+1} \triangleq \tfrac{\pi - \theta^+ - \phi_n}{\omega_n} + \tfrac{\theta^- + \phi_{n+1} + \pi}{\omega_{n+1}} \tag{5b}$$

one seeks a pair $(k_n, k_{n+1}) \in \mathbb{Z}_+^2$ for which

$$\tfrac{2\pi}{\omega_n} k_n - \tfrac{2\pi}{\omega_{n+1}} k_{n+1} \in [-\underline{g}_{n,n+1}, \overline{g}_{n,n+1}] \tag{6}$$

Proposition 1: Consider two oscillators n and $n+1$ with dynamics satisfying (2) and (3), having periods $\tau_n = 2\pi\omega_n^{-1}$ and $\tau_{n+1} = 2\pi\omega_{n+1}^{-1}$. The two oscillators will achieve rendezvous iff the region in the x-y plane

$$\{(x,y) \in \mathbb{R}^2 \mid \tau_n x - \tau_{n+1} y - \overline{g}_{n,n+1} \leq 0 \\ \wedge\, \tau_n x - \tau_{n+1} y + \underline{g}_{n,n+1} \geq 0\} \tag{7}$$

has a nonempty intersection with the lattice $\mathbb{Z}^2$.

Proof: For the necessity part assume rendezvous happens after k_n periods of oscillator n and k_{n+1} periods of oscillator $n+1$. Since $\left[\frac{2k_n\pi + \theta^+ - \phi_n}{\omega_n}, \frac{2(k_n+1)\pi - \theta^+ - \phi_n}{\omega_n}\right] \cap \left[\frac{2(k_{n+1}-1)\pi - \theta^- - \phi_{n+1}}{\omega_{n+1}}, \frac{2k_{n+1}\pi + \theta^- - \phi_{n+1}}{\omega_{n+1}}\right] \neq \varnothing$, it follows that (6) is satisfied. Condition (6), specifically takes the form $\tau_n k_n - \tau_{n+1} k_{n+1} \in [-\underline{g}_{n,n+1}, \overline{g}_{n,n+1}]$, implying $\tau_n k_n - \tau_{n+1} k_{n+1} + \underline{g}_{n,n+1} \geq 0$ *and* $\tau_n k_n - \tau_{n+1} k_{n+1} - \overline{g}_{n,n+1} \leq 0$. Sufficiency is shown tracing back the same steps: if there is an integer pair (k_n, k_{n+1}) such that $\tau_n k_n - \tau_{n+1} k_{n+1} + \underline{g}_{n,n+1} \geq 0$ and $\tau_n k_n - \tau_{n+1} k_{n+1} - \overline{g}_{n,n+1} \leq 0$, then $\tau_n k_n - \tau_{n+1} k_{n+1} \in [-\underline{g}_{n,n+1}, \overline{g}_{n+1}]$. This means that $T_n^+ \cap T_{n+1}^- \neq \varnothing$. Shortly after time crosses the lower boundary of this set, the two oscillators will rendezvous for the first time. ■

Proposition 1 does not inform about how to find the pair of period multiples that should be completed before rendezvous. Surprisingly, a number-theoretic approach may be more constructive. If the frequency ratio $\frac{\omega_n}{\omega_{n+1}}$ is *rational*, then there is always a positive real c such that both $c\frac{2\pi}{\omega_n} \equiv c\,\tau_n$ and $c\frac{2\pi}{\omega_{n+1}} \equiv c\,\tau_{n+1}$ are integers. Then (6) is written as $c\,\tau_n\, k_n - c\,\tau_{n+1}\, k_{n+1} \in [-c\,\underline{g}_{n,n+1}, c\,\overline{g}_{n,n+1}]$. If there exists an integer pair (k_n, k_{n+1}) to allow for rendezvous, then Proposition 1 implies that a linear *Diophantine equation*

$$c\,\tau_n\, k_n - c\,\tau_{n+1}\, k_{n+1} = \lambda \tag{8}$$

with a nonzero integer $\lambda \in [-c\,\underline{g}_{n,n+1}, c\,\overline{g}_{n,n+1}] \cap \mathbb{Z}$ has a solution. A necessary and sufficient condition [23] for this is λ being a multiple of the greatest common divider (gcd) of $c\,\tau_n$ and $c\,\tau_{n+1}$. In this case, the primitive solution of (8) can be obtained using the Euclidean algorithm [23].

Corollary 1: Assume $\frac{\omega_n}{\omega_{n+1}}$ is rational. Let $c \in \mathbb{R}$ be such that $\frac{2\pi c}{\omega_n} \in \mathbb{Z} \setminus \{0\} \ni \frac{2\pi c}{\omega_{n+1}}$. Then (2)–(3) achieve rendezvous iff there exists $\lambda \in [-c\,\underline{g}_{n,n+1}, c\,\overline{g}_{n,n+1}] \cap \mathbb{Z} \setminus \{0\}$ that is a multiple of $\gcd\left(\frac{2\pi c}{\omega_n}, \frac{2\pi c}{\omega_{n+1}}\right)$.

Corollary 1 suggests that there will definitely be an integer in the interval $[-c\,\underline{g}_{n,n+1}, c\,\overline{g}_{n,n+1}]$ if the length of this interval is larger than one. This result can also be arrived at from a different, more geometric, direction.

Indeed, without significant loss of generality, the (infinite) strip (7) can be bounded from above and below, assuming reasonably large bounds on the number of periods k_{max} one intends to wait for rendezvous to occur, and ignoring negative time based on (6) i.e. setting $k_{\mathsf{n,min}} \triangleq \frac{\theta^+ + \phi_n - \pi}{2\pi}$, $k_{\mathsf{n+1,min}} \triangleq \frac{-\theta^- + \phi_{n+1}}{2\pi}$. Then K can be made a convex body as follows:

$$K \triangleq \{(x,y) \in [k_{\mathsf{n,min}}, k_{\mathsf{max}}] \times [k_{\mathsf{n+1,min}}, k_{\mathsf{max}}] : \tau_n x - \tau_{n+1} y - \underline{g}_{n,n+1} \leq 0 \;\wedge\; \tau_n x - \tau_{n+1} y + \overline{g}_{n,n+1} \geq 0\} \quad (9)$$

If $\frac{\omega_n}{\omega_{n+1}}$ is rational, and $\gcd(c\,\tau_n, c\,\tau_{n+1}) = m$, then $a = \frac{c}{m}\tau_n$ and $b = \frac{c}{m}\tau_{n+1}$ will be coprime, in which case (9) is equivalent to

$$K = \{(x,y) \in [k_{\mathsf{n,min}}, k_{\mathsf{max}}] \times [k_{\mathsf{n+1,min}}, k_{\mathsf{max}}] : a\,x - b\,y - \tfrac{c\underline{g}_{n,n+1}}{m} \leq 0 \;\wedge\; a\,x - b\,y + \tfrac{c\overline{g}_{n,n+1}}{m} \geq 0\}$$

and once again, one would seek an integer λ in the interval $[-\frac{c\underline{g}_{n,n+1}}{m}, \frac{c\overline{g}_{n,n+1}}{m}]$. Now, however, Theorem 1 can be brought to bear, and offer an additional, geometric interpretation of the algebraic rendezvous condition formulated in Corollary 1.

The convex body K is a (bounded) strip between two parallel lines of slope $\frac{a}{b}$ at a distance $\frac{1}{\sqrt{a^2+b^2}}(-a, b) \cdot (0, \frac{c(\underline{g}_{n,n+1} + \overline{g}_{n,n+1})}{b\,m})^{\mathsf{T}} = \frac{c(\underline{g}_{n,n+1} + \overline{g}_{n,n+1})}{m\sqrt{a^2+b^2}}$ from each other. Since a and b are integers, $(-a, b)$ belongs in the dual lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2$. In addition, because a and b are coprime, $(-a, b)$ is primitive; in fact the vector for which the minimum is attained in (1):

$$\operatorname{width}(K, \mathbb{Z}^2) = \frac{c\,(\underline{g}_{n,n+1} + \overline{g}_{n,n+1})}{m} = \frac{c\,(\underline{g}_{n,n+1} + \overline{g}_{n,n+1})}{\gcd(c\,\tau_n, c\,\tau_{n+1})}$$

Due to a and b being coprime, the strip (9) is parallel to the lattice hyperplanes defined by the primitive vector (a, b). As a result, the widest strip that can "fit" in the planar lattice of $\mathbb{Z}^2$ without having lattice points in its interior, is one that in fact has two successive lattice hyperplanes themselves as boundaries. It is known that the maximal width function for empty lattice simplices (having their vertices on the the lattice) in $\mathbb{Z}^2$ is $f(2) = 1$ [24]. As a consequence of Theorem 1, therefore, any convex body K defined by (9) with lattice width larger than one, will inevitably have interior lattice points. The discussion thus traced its way to the following result:

Proposition 2: Assume $\frac{\tau_n}{\tau_{n+1}}$ is rational, let $c \in \mathbb{R}$ be the constant for which $c\,\tau_n \in \mathbb{Z} \setminus \{0\} \ni c\,\tau_{n+1}$, and set g_i according to (5). Then (2)–(3) achieve rendezvous if

$$\frac{c\,(\underline{g}_{n,n+1} + \overline{g}_{n,n+1})}{\gcd(c\,\tau_n, c\,\tau_{n+1})} > 1$$

The integer pairs (k_n, k_{n+1}) associated with rendezvous are themselves directly identified using Corollary 1. The time of the first rendezvous is pinpointed using the expression

$$t_{\mathsf{first}} = \max\{\tfrac{\theta_n^+ 0\phi_n + 2k_n\pi}{\omega_n}, \tfrac{2(k_{n+1}-1)\pi - \theta_{n+1}^- - \phi_{n+1}}{\omega_{n+1}}\}$$

B. Control strategy for synchronous rendezvous

According to the setup of Section III, the objective includes locking the agents in synchronous rendezvous. While the structure of the Diophantine equations suggests that once they meet once, they will continue to do so with some (possibly large) period, it is desirable to have *control* over the frequency of this rendezvous phenomenon, and potentially make this spatial conjunction longer and more robust. This can be done through short-term *control action*. This action has indeed to be short-term because in order to coordinate, the agents need to exchange information, and they can only do so while in rendezvous.

The first, and most drastic measure that the two oscillators are allowed to take is to reset their frequencies. Although several frequency setting schemes can be suggested, a simple strategy that ensures frequent subsequent synchronous rendezvous while distributing equally the effort involved in frequency reset is to average the oscillators' frequencies and adopt a common $\bar{\omega} = \frac{\omega_n + \omega_{n+1}}{2}$. When this happens, the two oscillators adopt identical dynamics with (3) becoming

$$\begin{bmatrix} \dot{x}_{1i} \\ \dot{x}_{2i} \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ -\bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} 0 \\ u_i \end{bmatrix} \qquad i \in \{n, n+1\}$$

The initial conditions of the two oscillators after the reset, however, are different. The task of regulating the states so that the time in rendezvous while oscillating with $\bar{\omega}$ is maximal, thus falls on u_i.

To devise the control strategy for u_i, first combine the two dynamics in one by defining the auxiliary variables. Let $o_{n,n+1}$ denote the centroid of agent n and $n+1$ rendezvous region; here $o_{n,n+1} \triangleq o_n + A = o_{n+1} - A$. Set

$$z_{1\,n,n+1} \triangleq x_{1\,n} + x_{1\,n+1} \qquad z_{2\,n,n+1} \triangleq x_{2\,n} + x_{2\,n+1}$$
$$u_{n,n+1} \triangleq u_n + u_{n+1}$$

and let $z_{n,n+1}$ denote the stack vector of $z_{1\,n,n+1}$ and $z_{2\,n,n+1}$. Note that $z_{1\,n,n+1} = 0$ implies that $\frac{p_n + p_{n+1}}{2} = o_{n,n+1}$, i.e. the (synchronized) oscillators are symmetrically distributed about the centroid of their rendezvous region. Since they have the same frequency, if their phases are off by π then they will be entering and leaving the rendezvous neighborhood at the same time, essentially maximizing the time they have to interact with each other. This motivates setting a control objective that aims at driving $z_{1\,n,n+1}$ to zero as quickly as possible: a time-optimal control problem.

The design and implementation of such a linear time-optimal controller is relatively straightforward [25], once the dynamics of $(z_{1\,n,n+1}, z_{2\,n,n+1})$ are written down. The details are skipped due to lack of space. As expected, the control law is of bang-bang form.

One implementation challenge stems from the fact that B being very small relative to A, the time that the optimal controller has to drive $z_{1\,n,n+1}$ to zero in a single rendezvous session may not be enough. The good news is that even when the optimal controller does not diminish $z_{1\,n,n+1}$ in one turn, the closer it brings it to zero, the more time it has to do so during the following rendezvous session, and it will be starting from a better initial condition.

Optimal control theory will thus dictate the combined control input $u_{n,n+1} = u_n + u_{n+1}$. For the first pair to rendezvous, one of the oscillators, say n, is arbitrarily labeled the leader, and its follower $n+1$, takes the responsibility of implementing the control law: $u_{n+1} = u_{n,n+1}$.

C. Extension to $N > 2$

Recall the original setup of Section III, where there were N oscillators arranged side-by-side on the real line, starting each with arbitrary phase and frequency. It is natural to ask if the synchronization strategy of Section IV-B can be leveraged to coordinate a string of oscillators.

The answer to this question is yes, with the introduction of a few additional coordination rules. One has to keep in mind that the oscillators cannot communicate with each other except when they find themselves in the same rendezvous; thus the possibility of propagating information for long-range coordination purposes is extremely limited.

Still, one can analyze the behavior of a string of very-low-range interacting oscillators under the control regime of Section IV-B. Under this scheme, the first pair of oscillators to achieve rendezvous —suppose for the sake of argument that this pair is $(n, n+1)$— will synchronize their frequencies and *commit* to oscillating at $\bar{\omega} = \frac{\omega_n + \omega_{n+1}}{2}$. Then the optimal control action drives $z_{1,n,n+1} \to 0$.

The strategy from this point on is to leave this pair to continue implementing the protocol of Section IV-B undisturbed. If any of the two coupled oscillators rendezvous with an *uncommitted* neighboring oscillator —suppose, again for the sake of argument, that this happens to be agent $n+2$— the latter is forced to *adot* the frequency of the group of already committed agents. The reason is that in the rendezvous region $[o_{n+2} - A - B, o_{n+2} - A + B]$ where $n+1$ and $n+2$ are now located, agent n is out of range and cannot synchronize with the new pair. Any deviation of $n+1$ from its committed frequency will desynchronize it with respect to n. The newly recruited agent $n+2$ has full responsibility to implement the optimal control law $u_{n+1,n+2}$: while in rendezvous within $[o_{n+2} - A - B, o_{n+2} - A + B]$, $u_{n+1} = 0$ and $u_{n+2} = u_{n+1,n+2}$.

Other uncommitted agents may spontaneously rendezvous in other parts of the line, and also implement the strategy of Section IV-B independently, and possibly start recruiting neighboring uncommitted agents, in the way described in the previous paragraph. When committed agents from two different groups encounter each other in a rendezvous region, they remain do not try to synchronize. Each agent remains committed and faithful to the group it originally joined.

It is relatively straightforward to track the evolution of group forming and frequency locking by iteratively solving a collection of ILP problems, either directly, or through the Diophantine equation (8). Originally, this collection involves all $N-1$ neighboring pairs of agents. The good news is that after each frequency locking, the collection is reduced by one: once an agent resets and commits to a frequency, it essentially becomes part of an existing oscillator, and the problem of predicting future rendezvous becomes progressively easier.

V. Validation

For illustration purposes, consider only three oscillators, of which the position before rendezvous is given by

$$p_1(t) = \sin(\omega_1 t + \phi_1) - 3 \,, \qquad p_2(t) = \sin(\omega_2 t + \phi_2) - 1$$
$$p_3(t) = \sin(\omega_3 t + \phi_3) + 1$$

The motion of the oscillators is parameterized by individual initial frequencies ω_i and phases ϕ_i for $i = 1, \ldots, 3$, which for this example are chosen arbitrarily:

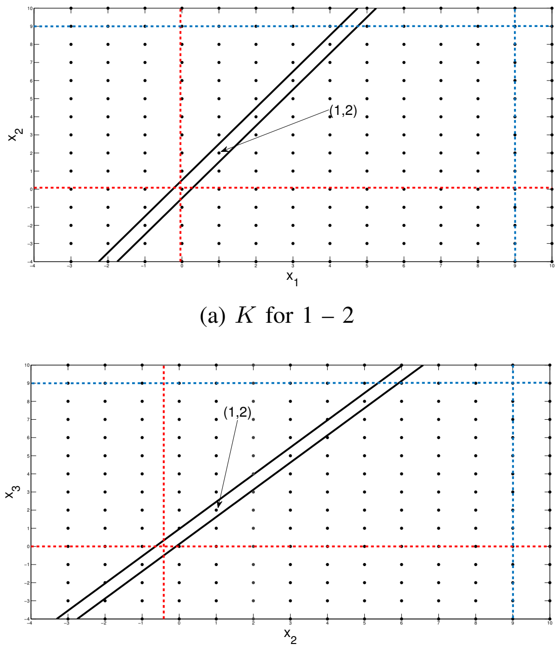
	agent 1	agent 2	agent 3
phase ϕ_i [rad]	$\frac{3\pi}{4}$	0	$-\frac{\pi}{6}$
frequency ω_i [rad/sec]	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$

Their linear arrangement allows rendezvous between agents 1 and 2, and between agents 2 and 3. Given parameters, the convex bodies for the two pairs are shown in Fig. 2.

Both convex bodies contain multiple lattice points, and the one of smallest distance from the origin is in both cases $(1, 2)$. For example, for Fig. 2a Corollary 1 with $\frac{c}{m} = \frac{1}{12}$ requires the existence of an integer in $[-\frac{1}{2}, -\frac{1}{2}]$, while for Fig. 2b the interval is $[-\frac{23}{12}, -\frac{1}{4}]$. The solutions suggest an earliest rendezvous time between 1 and 2 at 19 seconds, and for 2 and 3 at 13.33 seconds. It follows that 2 and 3 will pair first and lock their frequencies to $\bar{\omega} = \frac{\omega_2 + \omega_3}{2} = \frac{5\pi}{24}$, after which time agent 1 will be recruited and adopt this same frequency. Figure 3 illustrates the agents' oscillations and marks the rendezvous events with the brief bang-bang optimal control action, that together with the frequency resets makes the subsequent rendezvous events both regular and slightly longer lasting.

VI. Conclusion

The problem of synchronous rendezvous for oscillators with very-short-range interaction, appears to have strong connections with ILP and links to particular topics in number theory and the geometry of numbers. Exploiting standard results about the solutions of linear Diophantine equations and utilizing Khinchine's flatness theorem, one can derive conditions under which two neighboring oscillators that



(a) K for 1 – 2

(b) K for 2–3

Fig. 2: The convex bodies for the two pairs of oscillators which can rendezvous, indicating the nonzero integer multiple of periods for each member of the pair after which their first rendezvous occurs.

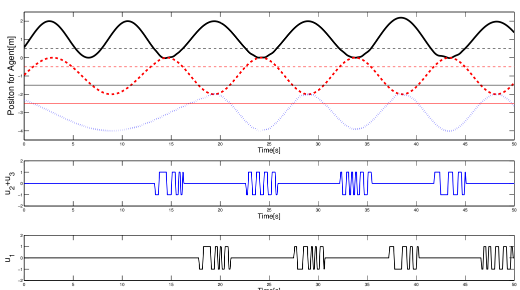


Fig. 3: Agent paths and control action during rendezvous.

evolve on the same line will rendezvous in a very small region neighboring their oscillation domain boundaries. When rendezvous occurs, optimal control theory can inform the design of controllers that progressively increase the time agents coincide within the small rendezvous regions.

Knowing where and when rendezvous happens is of interest in cases where one needs to develop visiting itineraries. Agents could for example be spatially distributed sensor platforms that roam and collect data, and because of limited local storage they need to be periodically upload those data via some low-range wireless channels. It is important for the vehicles tasked with harvesting these data to know at what time they need to be where.

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