Robotica (2001) volume 19, pp. 675–690. Printed in the United Kingdom

Mobile manipulator modeling with Kane's approach

Herbert G. Tanner and Kostas J. Kyriakopoulos

Control Systems Laboratory, Dept. of Mechanical Engineering, National Technical University of Athens, 9 Heroon Polytechniou St., 15 780 Zografou, Athens (GREECE)
E-mail: {htanner,kkyria}@central.ntua.gr

(Received in Final Form: February 8, 2001)

SUMMARY

A wheeled mobile manipulator system is modeled using Kane's dynamic equations. Kane's equations are constructed with minimum effort, are control oriented and provide both physical insight and fast simulations. The powerful tools of Kane's approach for incorporating nonholonomic motion constraints and bringing noncontributing forces into evidence are exploited. Both nonholonomic constraints associated with slipping and skidding as well as conditions for avoiding tipping over are included. The resulting equations, along with the set of constraint equations provide a safe and complete framework for developing control strategies for mobile manipulator systems.

KEYWORDS: Manipulator modeling; Kane's approach; Nonholonomic restraints; Control strategies.

1. INTRODUCTION

A mobile manipulator is a system composed of a manipulator attached to a mobile base. In this way, the manipulator workspace is drastically increased. Due to the motion constraints imposed on the mobile platform, such a system is usually nonholonomic.[1] Nonholonomic constraints reduce the dimension of the state space without affecting the system's configuration space.

Many of the models proposed for mobile manipulator systems[2,3] do not include nonholonomic constraints. In reference [2] an underwater vehicle with multiple robotic manipulators is modeled. Dynamic modeling is achieved by the use of Kane's dynamic equations.[4] Angular momentum preservation laws impose nonholonomic constraints on the URV which are not included in the model. Khatib et al.[3] model cooperating mobile manipulators, the mobile platforms of which move on a planar surface. They obtain the dynamic equations using the Lagrange formulation in operational space,[3] assuming that the mobile platforms can move in a holonomic way. Other models include nonholonomic constraints using Lagrange[5] and Newton-Euler formulations.[6] In these models not all kinds of nonholonomic constraints imposed are considered. Chen and Zalzala[6] include the nonholonomic constraint associated with the no skidding condition. In reference [5] the nonholonomic constraint resulting from the no slipping condition is also included. Constraints are merged with the system equations using Lagrange multipliers.

Recently, Thanjavur and Rajagopala[7] modeled an AGV using Kane equations. They pointed out the merits of using Kane's approach to model vehicles and utilized some of the tools to incorporate nonholonomy. They focused on the dynamics of the vehicle individual components (drive and castor wheels, drive wheel assemblies, etc.) and included the no slipping and no skidding constraints.

Modeling is usually treated as a preprocessing stage for the application of a control strategy. However, careful modeling can remove unnecessary detail and reveal some hidden characteristics of the system at hand. Thus, instead of being a mere system description, an appropriately constructed dynamic model can greatly facilitate and guide the control design.

In this paper a dynamic model is developed, following Kane's approach. Compared to Lagrange formulation, Kane's methodology involves less arithmetic operations and is thus simpler and faster in simulation.[2–4] What is more, Kane's equations can be easily brought into closed form[2] which is best suited for control purposes. The modeling procedure presented in this paper enables one to consider the mobile manipulator as a single system. With Kane's methodology it becomes clear that the dynamic interaction between the vehicle and the manipulator does not influence performance and thus need not be compensated. On the other hand the same interaction is brought into evidence since it is useful for ensuring contact stability with the ground. Additional dynamic constraints are provided, involving the ground reaction forces, which eventually yield a sufficient set of dynamic stability conditions. Dynamic effects on tippover stability are considered in reference [8]. This paper extends and enriches the early results of reference [9] by focusing on the contact conditions, exploiting the nonholonomy features of Kane's approach and emphasizing on the role of the interaction forces and torques between the vehicle and the manipulator in maintaining contact stability with the ground and finally investigating computational complexity issues.

The rest of the paper is organized as follows: In Section 2 we introduce the basic terminology used in the paper and state the assumptions made. In Section 3 we develop the dynamics of the mechanism and calculate the contributions of each of its components to the dynamic equations. Section 4 presents in dynamic form the constraints imposed on the system and in Section 5 issues of physical insight, computational complexity and comparison to alternative methodologies are discussed. A numerical example is presented in Section 6 and several simulation cases are analyzed. Finally, Section 7 concludes with the main points made in this paper.

2. PRELIMINARIES

Consider a fixed inertial frame, $\{I\}$ as in Figure 1. Its **x** and **y** axes lay on the horizontal plane. At the mobile platform mass center, a frame $\{v\}$ is attached. Axis $\mathbf{z}^v$ is parallel to $\mathbf{z}^I$. At this mass center, the central principal inertial frame, $\{v_{cpi}\}$, is assigned, which generally differs from $\{v\}$. The point where the manipulator is attached on the platform is labeled 0 and coincides with the platform point $\bar{0}$ (the distinction between the two points will be justified in the sequel). On each manipulator link the central principal inertial frame is identified. Finally, a frame is assigned at a fixed point of the manipulator end effector. The superscript on the right side of any quantity will denote the rigid body which it refers to. The superscript on the left will denote the reference frame with respect to which the quantity is expressed. When omitted, frame $\{I\}$ is supposed.

At the center of each wheel $j, j=1, \ldots, w$ of the vehicle, frame $\{w_j\}$ is attached (Figure 1). Its $\mathbf{z}^{w_j}$ axis remains parallel to the vehicle $\mathbf{z}^v$ axis, but the frame can rotate around this axis. Each frame is related to the fixed frame through a rotation matrix **R**. This rotation matrix will be written as, i.e. ${}^{I}_{w_j}\mathbf{R}$, to indicate transformation of free vectors expressed in frame $\{w_j\}$ to the inertial frame. A rotation matrix will be expressed as

$${}^{j}_{i}\mathbf{R} = \begin{bmatrix} \pi^{x^i}_{x^j} & \pi^{y^i}_{x^j} & \pi^{z^i}_{x^j} \\ \pi^{x^i}_{y^j} & \pi^{y^i}_{y^j} & \pi^{z^i}_{y^j} \\ \pi^{x^i}_{z^j} & \pi^{y^i}_{z^j} & \pi^{z^i}_{z^j} \end{bmatrix},\ \pi^{x^i}_{x^j} \text{ being the direction cosine between } \mathbf{x}^i \text{ and } \mathbf{x}^j \text{ and } (\pi^{n^i}_{n^j})_{q_j} \triangleq \frac{\partial \pi^{n^i}_{n^j}}{\partial q_j}$$

The coefficient of static friction between a wheel and the ground is assumed the same for all wheels and equal to η. When exploiting slipping phenomena, one has to consider the friction conditions between a wheel and the ground more systematically. Tire friction with relation to traction is an issue of ongoing research[10] and lies beyond the scope of this paper.

The manipulator is supposed to have $n-4=m$ rotational joints. Define the generalized coordinates as

$$\mathbf{q} \triangleq [x^v \ \ y^v \ \ \theta \ \ \phi \ \ q^1 \ \ \ldots \ \ q^m]^T$$

where x, y are the vehicle planar coordinates in the fixed coordinate system $\{I\}$, θ is its orientation, ϕ is the steering angle of the vehicle and the rest correspond to the manipulator joints. The number of generalized coordinates is n since the constraints imposed are completely nonholonomic.

Define the following generalized speeds:

$$u_3 \triangleq \dot{\theta} \qquad u_4 \triangleq \dot{\phi} \qquad u_r \triangleq \dot{q}^{r-4} \ \ r=5, \ldots, m+4$$

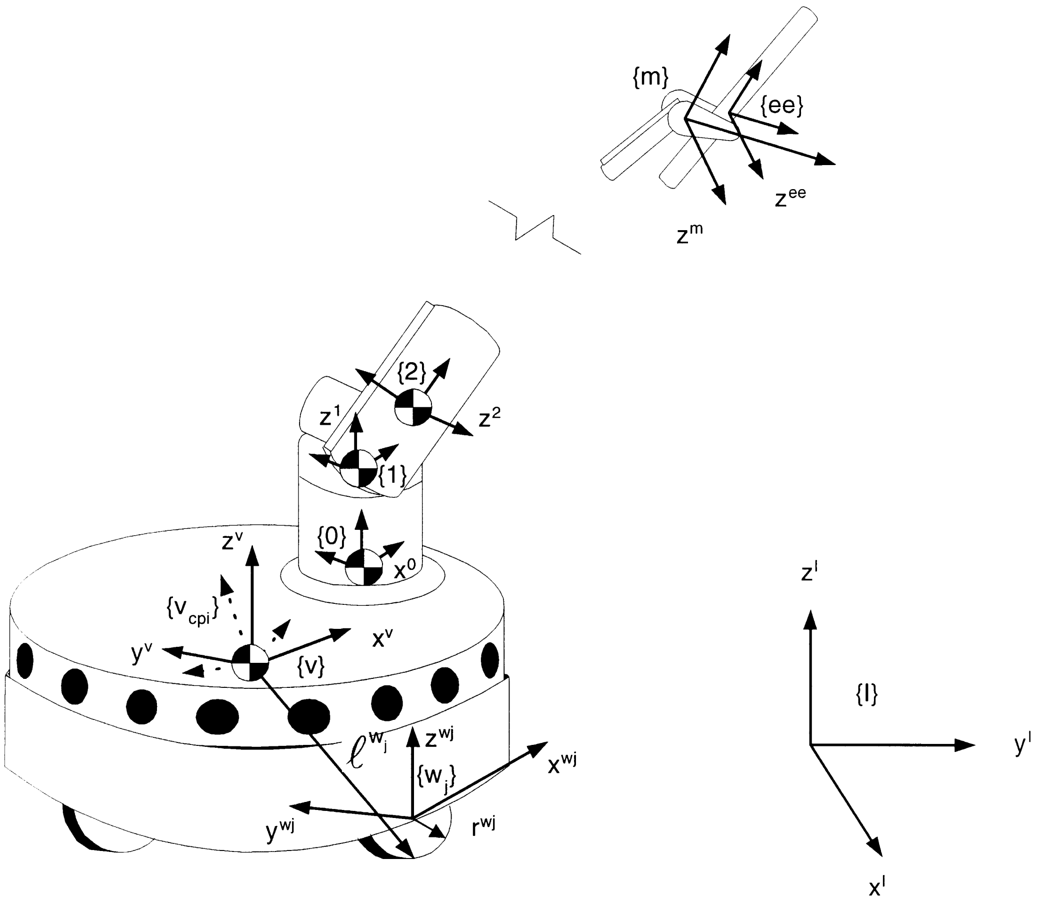

Fig. 1. Frame assignment on Mobile Manipulator.

In order to save writing and produce to efficient computer code, we introduce auxiliary functions ($\mathbb{Z}, \mathbb{X}, \mathbb{Y}$, etc.)[11] representing expressions appearing repeatedly. These auxiliary functions are defined in the Appendix.

In the sequel it is assumed that the motion of the vehicle is restricted to the horizontal plane. It is also assumed that each link on the manipulator rotates relatively to the previous link in the chain, only in a direction parallel to one of its principal axes of inertia, named z.

3. RIGID BODY CONTRIBUTIONS

In this section the kinematics and dynamics of each rigid body in the mechanism will be briefly discussed in order to determine the contribution of each body to the dynamics of the system. The detailed derivations have been omitted to preserve the more general perspective, however all dynamic contributions can be analytically calculated using the auxiliary terms given in the appendix.

3.1. Vehicle wheels

From Kane's approach it follows that all reaction forces exerted on the wheels make no contribution to dynamics. Once the nonholonomic motion constraints have been included, due to the motion being planar the ground can be simply ignored.

Each wheel is being modeled as a rotating disk. At the center of each wheel j a reference frame $\{w_j\}$ is attached. The axes are parallel to the central principal inertial axes of the wheel: axis $\mathbf{x}^{w_j}$ points towards the direction of the wheel linear velocity; $\mathbf{y}^{w_j}$ is the rolling axis and $\mathbf{z}^{w_j}$ is vertical. A wheel has two degrees of freedom. One allows rotation about y^{w_j} axis (Figure 1) and is controlled by an input torque $\tau_d^{w_j}$. The second degree of freedom corresponds to the steering angle of each wheel ϕ^{w_j}. Normally, all steering angles are coupled to achieve kinematic compatibility. The value of only one of them and the equations of the steering mechanism are sufficient to describe them all. Therefore, we can assume that $\phi^{w_j} = \phi^{w_j}(\phi)$, where ϕ is one of them, named the steering angle.

The velocity in $\{I\}$ of frame's $\{w_j\}$ origin can be expressed as

$$\mathbf{v}^{w_j} = \dot{x}^v \mathbf{x}^I + \dot{y}^v \mathbf{y}^I + \dot{\theta}(\mathbf{z}^v \times \boldsymbol{\ell}^{w_j}),$$

where $\boldsymbol{\ell}^{w_j}$ is the position vector from the origin of $\{v\}$ to the contact point of the wheel with the ground (Figure 1). The no skidding condition states that there should be no velocity component normal to the wheel plane. Expressing $\mathbf{v}^{w_j}$ w.r.t. the wheel frame this statement is equivalent to

$$\dot{\theta} = \frac{\dot{x}^v \sin(\phi^{w_j} + \theta) - \dot{y}^v \cos(\phi^{w_j} + \theta)}{{}^v\ell_y^{w_j} \sin \phi^{w_j} + {}^v\ell_x^{w_j} \cos \phi^{w_j}} \quad \text{for any } j \in \{1, \ldots, w\} \tag{1}$$

Under the no skidding condition the wheel velocity reduces to

$$\mathbf{v}^{w_j} = [\mathbb{Z}_{3_j} u_1 + \mathbb{Z}_{4j} u_2]\mathbf{x}^{w_j}, \tag{2}$$

while the *no slipping* constraint can be expressed as:

$$\mathbf{v}^{w_j} = \omega^{w_j} \times r^{w_j} \mathbf{z}^{w_j} \tag{3}$$

where r^{w_j} is the wheel radius.

Among the j wheels, identify the one which determines the steering direction and denote that angle by ϕ^o, so as to clearly distinguish it from the rest. The orientation angle of the vehicle will then be expressed in terms of that particular angle, according to (1): $\dot{\theta} = \frac{\dot{x}^v \sin(\phi^o + \theta) - \dot{y}^v \cos(\phi^o + \theta)}{{}^v\ell_y^{w_o} \sin \phi^{w_o} + {}^v\ell_x^{w_o} \cos \phi^{w_o}}$. Now let $u_1 \triangleq \frac{\dot{x}^v}{{}^v\ell_y^{w_o} \sin \phi^{w_o} + {}^v\ell_x^{w_o} \cos \phi^{w_o}}$ and $u_2 \triangleq \frac{\dot{y}^v}{{}^v\ell_y^{w_o} \sin \phi^{w_o} + {}^v\ell_x^{w_o} \cos \phi^{w_o}}$.

Then,

$$u_3 = \dot{\theta} = u_1 \sin(\phi^o + \theta) - u_2 \cos(\phi^o + \theta) \tag{4}$$

The angular velocity of the wheel is written:

$$\omega^{w_j} = \frac{1}{r^{w_j}} [\mathbb{Z}_{3j} u_1 + \mathbb{Z}_{4j} u_2]\mathbf{y}^{w_j} + [\mathbb{Z}_3 + \mathbb{Z}_{2j} u_4]\mathbf{z}^{w_j} \tag{5}$$

On each wheel the following contact and field forces (and torques) are exerted: the reaction from the ground, $\mathbf{F}^{w_j}$, applied at the point of contact between the wheel and the ground, the reaction from the vehicle body $\mathbf{F}^{vj}$, the control input torques $\mathbf{T}^{w_j}$, both applied at frame's $\{w_j\}$ origin and the inertial forces and torques. The *ground reactions* and the vehicle-wheel *interaction forces*, do not contribute to the dynamic equations.[11] The only contributing generalized forces are the *input torques* and the *inertial forces and torques*.

By setting $\alpha_y^{w_j} \triangleq \mathbb{Z}_{3j}\dot{u}_1 + \mathbb{Z}_{4j}\dot{u}_2 + \mathbb{Z}_{(\kappa+2)_j}$ and $\alpha_z^{w_j} \triangleq \mathbb{Z}_1\dot{u}_1 + \mathbb{Z}_2\dot{u}_2 + \mathbb{Z}_{2j}\dot{u}_4 + \mathbb{Z}_{(\kappa+3)_j}$, the generalized inertia and active forces which contribute to the dynamic equations can be expressed:

$$T_1^{*w_j} = -\alpha_y^{w_j}\mathbb{Z}_{3j}\frac{m^{w_j}}{2} - \alpha_z^{w_j}\mathbb{Z}_1\frac{m^{w_j}(r^{w_j})^2}{4}, \qquad T_2^{*w_j} = -\alpha_y^{w_j}\mathbb{Z}_{4j}\frac{m^{w_j}}{2} - \alpha_z^{w_j}\mathbb{Z}_2\frac{m^{w_j}(r^{w_j})^2}{4},$$

$$T_4^{*w_j} = -\alpha_z^{w_j}\mathbb{Z}_{2j}\frac{m^{w_j}(r^{w_j})^2}{4}, \qquad F_1^{*w_j} = -m^{w_j}\alpha_y^{w_j}\mathbb{Z}_{3j}, \qquad F_2^{*w_j} = -m^{w_j}\alpha_y^{w_j}\mathbb{Z}_{4j}$$

The contributions of the input and inertial forces and torques to the generalized active forces are

$$(T_s^{w_j})_1 = \mathbb{Z}_1\tau_s^{w_j}, \quad (T_s^{w_j})_2 = \mathbb{Z}_2\tau_s^{w_j}, \quad (T_s^{w_j})_4 = \mathbb{Z}_{2j}\tau_s^{w_j}, \quad (T_d^{w_j})_1 = \frac{\mathbb{Z}_{3j}}{r^{w_j}}\tau_d^{w_j}, \quad (T_d^{w_j})_2 = \frac{\mathbb{Z}_{4j}}{r^{w_j}}\tau_d^{w_j},$$

where $\tau_s^{w_j}$ and $\tau_d^{w_j}$ are the steering and driving torques exerted by the motors on the wheels, respectively.

3.2. Vehicle body

The vehicle body moves horizontally in a nonholonomic fashion, subject to the combined effect of the forces exerted by the wheels and the forces exerted by the attached manipulator. The wheel forces are included in the form of reactions to the steering and traction torques exerted on the wheels.

At the mass center of the vehicle body a reference frame $\{v\}$ is attached. Axis $\mathbf{x}^v$ is aligned with the direction of motion when the steering angle is zero. Axis $\mathbf{z}^v$ is vertical and $\mathbf{y}^v$ completes the coordinate system. On the vehicle body the following contact and field forces and torques arc exerted: the reaction forces by the wheels, a force and torque by the manipulator, the vehicle weight, the inertial forces and torques and the reactions to the driving and steering torques. Among them, the reaction forces by the wheels, the force and torque applied by the manipulator and the weight of the vehicle do not contribute to the generalized active forces.

Let I^v be the vehicle's inertia moment tensor and define:

$$T_x^{*v} \triangleq (\mathbb{Z}_1\dot{u}_1 + \mathbb{Z}_2\dot{u}_2 + \mathbb{Z}_{\kappa+1})\pi_{x^v cpi}^{z^v} I_x^v - (\mathbb{Z}_3)^2\pi_{y^v cpi}^{z^v}\pi_{z^v cpi}^{z^v}(I_y^v - I_z^v)$$
$$T_y^{*v} \triangleq (\mathbb{Z}_1\dot{u}_1 + \mathbb{Z}_2\dot{u}_2 + \mathbb{Z}_{\kappa+1})\pi_{y^v cpi}^{z^v} I_y^v - (\mathbb{Z}_3)^2\pi_{z^v cpi}^{z^v}\pi_{x^v cpi}^{z^v}(I_z^v - I_x^v)$$
$$T_z^{*v} \triangleq (\mathbb{Z}_1\dot{u}_1 + \mathbb{Z}_2\dot{u}_2 + \mathbb{Z}_{\kappa+1})\pi_{z^v cpi}^{z^v} I_z^v - (\mathbb{Z}_3)^2\pi_{x^v cpi}^{z^v}\pi_{y^v cpi}^{z^v}(I_x^v - I_y^v)$$

Let the reaction torques by the wheels and the interaction forces exerted by the manipulator be denoted, respectively,

$$T_s^{\overline{w_j}} = -\tau_s^{w_j}\mathbf{z}^{w_j} \quad T_d^{\overline{w_j}} = -\tau_d^{w_j}\mathbf{y}^{w_j} \quad \mathbf{F}^{\bar{0}} = F_x^{\bar{0}}\mathbf{x}^v + F_y^{\bar{0}}\mathbf{y}^v + F_z^{\bar{0}}\mathbf{z}^v \quad \mathbf{T}^{\bar{0}} = T_x^{\bar{0}}\mathbf{x}^v + T_y^{\bar{0}}\mathbf{y}^v + T_z^{\bar{0}}\mathbf{z}^v$$

Then contribution of the mobile platform to the generalized inertial forces can be written

$$T_1^{*v} = -T_x^{*v}\pi_{z^v}^{x^v cpi}\mathbb{Z}_1 - T_y^{*v}\pi_{z^v}^{y^v cpi}\mathbb{Z}_1 - T_z^{*v}\pi_{z^v}^{z^v cpi}\mathbb{Z}_1$$
$$T_2^{*v} = -T_x^{*v}\pi_{z^v}^{x^v cpi}\mathbb{Z}_2 - T_y^{*v}\pi_{z^v}^{y^v cpi}\mathbb{Z}_2 - T_z^{*v}\pi_{z^v}^{z^v cpi}\mathbb{Z}_2$$
$$F_1^{*v} = -m^v(\mathbb{Z}_o\dot{u}_1 + \mathbb{Z}_{\xi+1})\mathbb{Z}_o$$
$$F_2^{*v} = -m^v(\mathbb{Z}_o\dot{u}_2 + \mathbb{Z}_{\xi+2})\mathbb{Z}_o$$

and the contribution to the generalized active forces

$$(T_s^{\overline{w_j}})_1 = -\mathbb{Z}_1\tau_s^{w_j} \qquad (T_s^{\overline{w_j}})_2 = -\mathbb{Z}_2\tau_s^{w_j} \qquad T_1^{\bar{0}} = T_z^{\bar{0}}\mathbb{Z}_1 \quad T_2^{\bar{0}} = T_z^{\bar{0}}\mathbb{Z}_2$$
$$F_1^{\bar{0}} = F_x^{\bar{0}}\mathbb{Z}_{\lambda+3} + F_y^{\bar{0}}\mathbb{Z}_{\lambda+5} \qquad F_2^{\bar{0}} = F_x^{\bar{0}}\mathbb{Z}_{\lambda+4} + F_y^{\bar{0}}\mathbb{Z}_{\lambda+6}$$

3.3. MANIPULATOR BASE

The manipulator base is rigidly attached to the mobile platform. It does not possess an independent degree of freedom and could be considered a part of the mobile platform. The reason for examining it separately, despite computational cost, is the fact that for stationary manipulators the inertial parameters of their base are typically ignored and are usually unavailable. On the other hand, mobile platforms are manufactured separately. As a result their inertial characteristics include only the body and the wheels. The unavailability of information that would enable the integration of the manipulator base to other structural components necessitates its independent study.

The interaction forces between the manipulator base and the vehicle on which it is attached to do not contribute to the dynamics and therefore do not appear in the equations. However, these forces are useful in determining and maintaining contact stability of the wheels. In order to bring into evidence, the interaction forces between the manipulator and the platform we assume that there is a virtual relative movement between the two bodies, expressed in terms of some additional fictitious generalized speeds. At the mass center of the manipulator base, c_0, a reference frame {0} is attached. This frame is aligned with frame $\{v\}$.

On the manipulator base the following forces and torques are exerted: inertial forces and moments, reactions forces by the mobile platform, the reaction torque by the first link of the manipulator, and the base weight. The *base weight* contributes because of the virtual motion of base with respect to the platform.

Let I^0 be the base's inertia moment tensor and define:

$$F_x^{*0} \triangleq \mathbb{Z}_{\lambda+3}\dot{u}_1 + \mathbb{Z}_{\lambda+4}\dot{u}_2 + \mathbb{Z}_{\xi+3}, \qquad F_y^{*0} \triangleq \mathbb{Z}_{\lambda+5}\dot{u}_1 + \mathbb{Z}_{\lambda+6}\dot{u}_2 + \mathbb{Z}_{\xi+4}$$

$$T_x^{*0} \triangleq (\mathbb{Z}_1\dot{u}_1 + \mathbb{Z}_2\dot{u}_2 + \mathbb{Z}_{\kappa+1})\pi_{x^{0cpi}}^{z^v} I_x^0 - (\mathbb{Z}_3)^2 \pi_{y^{0cpi}}^{z^v}\, \pi_{z^{0cpi}}^{z^v} (I_y^0 - I_z^0)$$

$$T_y^{*0} \triangleq (\mathbb{Z}_1\dot{u}_1 + \mathbb{Z}_2\dot{u}_2 + \mathbb{Z}_{\kappa+1})\pi_{y^{0cpi}}^{z^v} I_y^0 - (\mathbb{Z}_3)^2 \pi_{z^{0cpi}}^{z^v}\, \pi_{x^{0cpi}}^{z^v} (I_z^0 - U_x^0)$$

$$T_z^{*0} \triangleq (\mathbb{Z}_1\dot{u}_1 + \mathbb{Z}_2\dot{u}_2 + \mathbb{Z}_{\kappa+1})\pi_{z^{0cpi}}^{z^v} I_z^0 - (\mathbb{Z}_3)^2 \pi_{x^{0cpi}}^{z^v}\, \pi_{y^{0cpi}}^{z^v} (I_x^0 - U_y^0)$$

The interaction forces and torques exerted by the platform to the base are expressed as

$$\mathbf{F}^0 = -F_x^{\bar{0}}\mathbf{x}^v - F_y^{\bar{0}}\mathbf{y}^v - F_z^{\bar{0}}\mathbf{z}^v \qquad \mathbf{T}^0 = -T_x^{\bar{0}}\mathbf{x}^v - T_y^{\bar{0}}\mathbf{y}^v - T_z^{\bar{0}}\mathbf{z}^v$$

whereas the reaction torque by link 1 and the weight is $\mathbf{T}^{\bar{1}} = -\tau_1\mathbf{z}^1$ and $\mathbf{G}^0 = m^0 g\mathbf{z}^I$, respectively.

The contribution of the inertial moment to the generalized inertial forces will be

$$T_1^{*0} = -T_x^{*0}\mathbb{Z}_1\pi_{x^{0cpi}}^{z^v} - T_y^{*0}\mathbb{Z}_1\pi_{y^{0cpi}}^{z^v} - T_z^{*0}\mathbb{Z}_1\pi_{z^{0cpi}}^{z^v} \qquad T_2^{*0} = -T_x^{*0}\mathbb{Z}_2\pi_{x^{0cpi}}^{z^v} - T_y^{*0}\mathbb{Z}_2\pi_{y^{0cpi}}^{z^v} - T_z^{*0}\mathbb{Z}_2\pi_{z^{0cpi}}^{z^v}$$

$$T_{m+5}^{*0} = -T_x^{*0}\pi_{x^v}^{x^{0cpi}} - T_y^{*0}\pi_{x^v}^{y^{0cpi}} - T_z^{*0}\pi_{x^v}^{z^{0cpi}} \qquad T_{m+6}^{*0} = -T_x^{*0}\pi_{y^v}^{x^{0cpi}} - T_y^{*0}\pi_{y^v}^{y^{0cpi}} - T_z^{*0}\pi_{y^v}^{z^{0cpi}}$$

$$T_{m+7}^{*0} = -T_x^{*0}\pi_{z^v}^{x^{0cpi}} - T_y^{*0}\pi_{y^v}^{y^{0cpi}} - T_z^{*0}\pi_{z^v}^{z^{0cpi}}$$

The contribution of the inertial force to the generalized inertial forces will accordingly be

$$F_1^{*0} = -m^0[F_x^{*0}\mathbb{Z}_{\lambda+3} + F_y^{*0}\mathbb{Z}_{\lambda+5}] \qquad F_2^{*0} = -m^0[F_x^{*0}\mathbb{Z}_{\lambda+4} + F_y^{*0}\mathbb{Z}_{\lambda+6}]$$

$$F_{m+8}^{*0} = -m^0 F_x^{*0} \qquad F_{m+9}^{*0} = -m^0 F_y^{*0}$$

The base weight, the interaction forces and the reaction to the torque on link 1 contribute by:

$$G_{m+10}^0 = m^0 g \qquad F_{m+8}^0 = -F_x^{\bar{0}} \qquad F_{m+9}^0 = -F_y^{\bar{0}} \qquad F_{m+10}^0 = -F_z^{\bar{0}}$$

$$T_1^0 = -T_z^{\bar{0}}\mathbb{Z}_1 \qquad T_2^0 = -T_z^{\bar{0}}\mathbb{Z}_2 \qquad T_{m+5}^0 = -T_x^{\bar{0}} \qquad T_{m+6}^0 = -T_y^{\bar{0}}$$

$$T_{m+7}^0 = -T_z^{\bar{0}} \qquad T_1^{\bar{1}} = -\tau_1\pi_{z^1}^{z^v}\mathbb{Z}_1 \qquad T_2^{\bar{1}} = -\tau_1\pi_{z^1}^{z^v}\mathbb{Z}_2 \qquad T_{m+5}^{\bar{1}} = -\tau_1\pi_{z^1}^{x^v}$$

$$T_{m+6}^{\bar{1}} = -\tau_1\pi_{z^1}^{y^v} \qquad T_{m+7}^{\bar{1}} = -\tau_1\pi_{z^1}^{x^v} \qquad F_1^0 = -F_x^{\bar{0}}\mathbb{Z}_{\lambda+3} - F_y^{\bar{0}}\mathbb{Z}_{\lambda+5} \qquad F_2^0 = -F_x^{\bar{0}}\mathbb{Z}_{\lambda+4} - F_y^{\bar{0}}\mathbb{Z}_{\lambda+6}$$

3.4. First manipulator link

Within Kane's methodology it is pointed out that out of all forces and torques acting on a link, only the input torque, the weight and the inertial forces and moments contribute. The reaction forces and torques between neighboring links do not contribute to dynamics and can be ignored. This observation results in significant savings in terms of computational and derivation complexity.

Frame {1} is attached to the mass center of the first manipulator link. Aligned to the principal inertial frarne, it rotates relative to the manipulator base around axis $\mathbf{z}^1$.

On the first link of the manipulator the following are applied: inertial forces and moments, the link weight, the input torque and the reaction to the input torque applied to the next link.

Define the terms:

$$F_x^{*1} \triangleq \mathbb{Z}_{\lambda+9}\dot{u}_1 + \mathbb{Z}_{\lambda+10}\dot{u}_2 + \mathbb{Z}_{\lambda+11}\dot{u}_5 + \mathbb{Z}_{\xi+5} \qquad T_x^{*1} \triangleq (\mathbb{Z}_4\dot{u}_1 + \mathbb{Z}_5\dot{u}_2 + \mathbb{Z}_{\kappa+4})I_x^1 - \mathbb{Z}_{11}\mathbb{Z}_{12}(I_y^1 - I_z^1)$$

$$F_y^{*1} \triangleq \mathbb{Z}_{\lambda+12}\dot{u}_1 + \mathbb{Z}_{\lambda+13}\dot{u}_2 + \mathbb{Z}_{\lambda+14}\dot{u}_5 + \mathbb{Z}_{\xi+6} \qquad T_y^{*1} \triangleq (\mathbb{Z}_6\dot{u}_1 + \mathbb{Z}_7\dot{u}_2 + \mathbb{Z}_{\kappa+5})I_y^1 - \mathbb{Z}_{12}\mathbb{Z}_{10}(I_z^1 - I_x^1)$$

$$F_z^{*1} \triangleq \mathbb{Z}_{\lambda+15}\dot{u}_1 + \mathbb{Z}_{\lambda+16}\dot{u}_2 + \mathbb{Z}_{\xi+7} \qquad T_z^{*1} \triangleq (\mathbb{Z}_8\dot{u}_1 + \mathbb{Z}_9\dot{u}_2 + \dot{u}_5 + \mathbb{Z}_{\kappa+6})I_z^1 - \mathbb{Z}_{10}\mathbb{Z}_{11}(I_x^1 - I_y^1)$$

The input torque of link 1 and the link's weight are, respectively, $\mathbf{T}^1 = \tau_1\mathbf{z}^1$ and $\mathbf{G}^1 = m^1 g\mathbf{z}^i$.

The contribution to the generalized inertial forces for link 1 would be

$$
\begin{aligned}
&F^{*1}_{1} = -m^1[F^{*1}_{x}\mathbb{Z}_{\lambda+9} + F^{*1}_{y}\mathbb{Z}_{\lambda+12} + F^{*1}_{z}\mathbb{Z}_{\lambda+15}] && T^{*1}_{1} = -T^{*1}_{x}\mathbb{Z}_{4} - T^{*1}_{y}\mathbb{Z}_{6} - T^{*1}_{z}\mathbb{Z}_{8}\\
&F^{*1}_{2} = -m^1[F^{*1}_{x}\mathbb{Z}_{\lambda+10} + F^{*1}_{y}\mathbb{Z}_{\lambda+13} + F^{*1}_{z}\mathbb{Z}_{\lambda+16}] && T^{*1}_{2} = -T^{*1}_{x}\mathbb{Z}_{5} - T^{*1}_{y}\mathbb{Z}_{7} - T^{*1}_{z}\mathbb{Z}_{9}\\
&F^{*1}_{5} = -m^1[F^{**1}_{x}\mathbb{Z}_{\lambda+11} + F^{*1}_{y}\mathbb{Z}_{\lambda+14}] && T^{*1}_{5} = -T^{*1}_{z}\\
&F^{*1}_{m+5} = -m^1[F^{**1}_{x}\mathbb{Z}_{\nu+1} + F^{*1}_{y}\mathbb{Z}_{\nu+7} + F^{*1}_{z}\mathbb{Z}_{\nu+13}] && T^{*1}_{m+5} = -T^{*1}_{x}\mathbb{Z}_{\mu+1} - T^{*1}_{y}\mathbb{Z}_{\mu+4} - T^{*1}_{z}\mathbb{Z}_{\mu+7}\\
&F^{*1}_{m+6} = -m^1[F^{**1}_{x}\mathbb{Z}_{\nu+2} + F^{*1}_{y}\mathbb{Z}_{\nu+8} + F^{*1}_{z}\mathbb{Z}_{\nu+14}] && T^{*1}_{m+6} = -T^{*1}_{x}\mathbb{Z}_{\mu+2} - T^{*1}_{y}\mathbb{Z}_{\mu+5} - T^{*1}_{z}\mathbb{Z}_{\mu+8}\\
&F^{*1}_{m+7} = -m^1[F^{**1}_{x}\mathbb{Z}_{\nu+3} + F^{*1}_{y}\mathbb{Z}_{\nu+9} + F^{*1}_{z}\mathbb{Z}_{\nu+15}] && T^{*1}_{m+7} = -T^{*1}_{x}\mathbb{Z}_{\mu+3} - T^{*1}_{y}\mathbb{Z}_{\mu+6} - T^{*1}_{z}\mathbb{Z}_{\mu+9}\\
&F^{*1}_{m+8} = -m^1[F^{**1}_{x}\mathbb{Z}_{\nu+4} + F^{*1}_{y}\mathbb{Z}_{\nu+10} + F^{*1}_{z}\mathbb{Z}_{\nu+16}] && T^{*1}_{m+9} = -m^1[F^{*1}_{x}\mathbb{Z}_{\nu+5} + F^{*1}_{y}\mathbb{Z}_{\nu+11} + F^{*1}_{z}\mathbb{Z}_{\nu+17}]\\
&F^{*1}_{m+10} = -m^1[F^{**1}_{x}\mathbb{Z}_{\nu+6} + F^{*1}_{y}\mathbb{Z}_{\nu+12} + F^{*1}_{z}\mathbb{Z}_{\nu+18}]
\end{aligned}
$$

On the other hand, the contribution of the weight is calculated as follows:

$$
\begin{aligned}
&G^1_1 = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\lambda+9} + \pi^{y^1}_{z^I}\mathbb{Z}_{\lambda+12} + \pi^{z^1}_{z^I}\mathbb{Z}_{\lambda+15}) && G^1_2 = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\lambda+10} + \pi^{y^1}_{z^I}\mathbb{Z}_{\lambda+13} + \pi^{z^1}_{z^I}\mathbb{Z}_{\lambda+16})\\
&G^1_5 = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\lambda+11} + \pi^{y^1}_{z^I}\mathbb{Z}_{\lambda+14}) && G^1_{m+5} = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\nu+1} + \pi^{y^1}_{z^I}\mathbb{Z}_{\nu+7} + \pi^{z^1}_{z^I}\mathbb{Z}_{\nu+13})\\
&G^1_{m+6} = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\nu+2} + \pi^{y^1}_{z^I}\mathbb{Z}_{\nu+8} + \pi^{z^1}_{z^I}\mathbb{Z}_{\nu+14}) && G^1_{m+7} = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\nu+3} + \pi^{y^1}_{z^I}\mathbb{Z}_{\nu+9} + \pi^{z^1}_{z^I}\mathbb{Z}_{\nu+15})\\
&G^1_{m+8} = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\nu+4} + \pi^{y^1}_{z^I}\mathbb{Z}_{\nu+10} + \pi^{z^1}_{z^I}\mathbb{Z}_{\nu+16}) && G^1_{m+9} = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\nu+5} + \pi^{y^1}_{z^I}\mathbb{Z}_{\nu+11} + \pi^{z^1}_{z^I}\mathbb{Z}_{\nu+17})\\
&G^1_{m+10} = m^1 g(\pi^{x^1}_{z^I}\mathbb{Z}_{\nu+6} + \pi^{y^1}_{z^I}\mathbb{Z}_{\nu+12} + \pi^{z^1}_{z^I}\mathbb{Z}_{\nu+18})
\end{aligned}
$$

while the input torque contributes by

$$
T^1_1 = \tau_1\mathbb{Z}_8 \qquad T^1_2 = \tau_1\mathbb{Z}_9 \qquad T^1_5 = \tau_1 \qquad T^1_{m+5} = \tau_1\,\pi^{x^\nu}_{z^1} \qquad T^1_{m+6} = \tau_1\,\pi^{y^\nu}_{z^1} \qquad T^1_{m+7} = \tau_1\,\pi^{x^\nu}_{z^1}
$$

3.5. Links 2 to m

Links 2 to m are analyzed as link 1 and recursive expressions for the kinematic quantities are derived. In this way, there is no constraint to the number of the manipulator joints that can be included in the model.

At the mass center of link i, for $i=2, \ldots, m$ frame $\{i\}$ is attached. It is aligned to the principal inertial frame. Link i rotates w.r.t. link $i-1$ along axis $\mathbf{z}^i$.

For the rest of the links the treatment is similar. We first identify the forces and torques acting on each link: the inertial forces and moments, the link's weight and the input torque. Define:

$$
F^{*i}_x = \mathbb{Z}_{\beta_i}\dot{u}_1 + \mathbb{Z}_{\beta_i+1}\dot{u}_2 + \mathbb{Z}_{\beta_i+2}\dot{u}_5 + \cdots + \mathbb{Z}_{\beta_i+i}\dot{u}_{i+3} + \mathbb{Z}_{\beta_i+i+1}\dot{u}_{i+4} + \mathbb{Z}_{\delta_i+1}
$$

$$
F^{*i}_y = \mathbb{Z}_{\beta_i+i+2}\dot{u}_1 + \mathbb{Z}_{\beta_i+i+3}\dot{u}_2 + \mathbb{Z}_{\beta_i+i+4}\dot{u}_5 + \cdots + \mathbb{Z}_{\beta_i+2i+2}\dot{u}_{i+3} + \mathbb{Z}_{\beta_i+2i+3}\dot{u}_{i+4} + \mathbb{Z}_{\delta_i+2}
$$

$$
F^{*i}_z = \mathbb{Z}_{\beta_i+2i+4}\dot{u}_1 + \mathbb{Z}_{\beta_i+2i+5}\dot{u}_2 + \mathbb{Z}_{\beta_i+2i+6}\dot{u}_5 + \cdots + \mathbb{Z}_{\beta_i+3i+4}\dot{u}_{i+3} + \mathbb{Z}_{\delta_i+3}
$$

$$
T^{*i}_x = (\mathbb{Z}_{\alpha_i}\dot{u}_1 + \mathbb{Z}_{\alpha_i+1}\dot{u}_2 + \mathbb{Z}_{\alpha_i+2}\dot{u}_5 + \cdots + \mathbb{Z}_{\alpha_i+1}\dot{u}_{i+3} + \mathbb{Z}_{\gamma_i+1})I^i_x - \mathbb{Z}_{\alpha_i+3i+4}\mathbb{Z}_{\alpha_i+3i+5}(I^i_y - I^i_z)
$$

$$
T^{*i}_y = (\mathbb{Z}_{\alpha_i+i+1}\dot{u}_1 + \mathbb{Z}_{\alpha_i+i+2}\dot{u}_2 + \mathbb{Z}_{\alpha_i+i+3}\dot{u}_5 + \cdots + \mathbb{Z}_{\alpha_i+2i+1}\dot{u}_{i+3} + \mathbb{Z}_{\gamma_i+2})I^i_y - \mathbb{Z}_{\alpha_i+3i+5}\mathbb{Z}_{\alpha_i+3i+3}(I^i_z - I^i_x)
$$

$$
T^{*i}_z = (\mathbb{Z}_{\alpha_i+2i+2}\dot{u}_1 + \mathbb{Z}_{\alpha_i+2i+3}\dot{u}_2 + \mathbb{Z}_{\alpha_i+2i+4}\dot{u}_5 + \cdots + \mathbb{Z}_{\alpha_i+3i+2}\dot{u}_{i+3} + \dot{u}_{i+4} + \mathbb{Z}_{\gamma_i+3})I^i_z - \mathbb{Z}_{\alpha_i+3i+3}\mathbb{Z}_{\alpha_i+3i+4}(I^i_x - I^i_y)
$$

The input torque and the weight of the link are $\mathbf{T}^i = \tau_i\mathbf{z}^i$ and $\mathbf{G}^i = m^i g\mathbf{z}^I$, respectively.

The contributions of the inertia moments and forces to the generalized inertial forces are:

$$
[T^{*i}_1 \quad T^{*i}_2 \quad T^{*i}_5 \quad \ldots \quad T^{*i}_{i+3} \quad T^{*i}_{i+4} \quad T^{*i}_{m+5} \quad T^{*i}_{m+6} \quad T^{*i}_{m+7}]^T
$$

$$
= -\begin{bmatrix}
\mathbb{Z}_{\alpha_i} & \mathbb{Z}_{\alpha_i+1} & \mathbb{Z}_{\alpha_i+2} & \ldots & \mathbb{Z}_{\alpha_i+i} & 0 & \mathbb{Z}_{\mu_i+1} & \mathbb{Z}_{\mu_i+2} & \mathbb{Z}_{\mu_i+3}\\
\mathbb{Z}_{\alpha_i+i+1} & \mathbb{Z}_{\alpha_i+i+2} & \mathbb{Z}_{\alpha_i+i+3} & \ldots & \mathbb{Z}_{\alpha_i+2i+1} & 0 & \mathbb{Z}_{\mu_i+4} & \mathbb{Z}_{\mu_i+5} & \mathbb{Z}_{\mu_i+6}\\
\mathbb{Z}_{\alpha_i+2i+2} & \mathbb{Z}_{\alpha_i+2i+3} & \mathbb{Z}_{\alpha_i+2i+4} & \ldots & \mathbb{Z}_{\alpha_i+3i+2} & 1 & \mathbb{Z}_{\mu_i+7} & \mathbb{Z}_{\mu_i+8} & \mathbb{Z}_{\mu_i+9}
\end{bmatrix}^T
\begin{bmatrix} T^{*i}_x \\ T^{*i}_y \\ T^{*i}_z \end{bmatrix},
$$

$$[F_1^{*i} \quad F_2^{*i} \quad F_5^{*i} \quad \ldots \quad F_{i+3}^{*i} \quad F_{i+4}^{*i} \quad F_{m+5}^{*i} \quad F_{m+6}^{*i} \quad F_{m+7}^{*i}]^T$$
$$= -m^i \begin{bmatrix} \mathbb{Z}_{\beta_i} & \mathbb{Z}_{\beta_i+1} & \mathbb{Z}_{\beta_i+2} & \ldots & \mathbb{Z}_{\beta_i+i} & \mathbb{Z}_{\beta_i+i+1} & \mathbb{Z}_{\nu_i+1} & \mathbb{Z}_{\nu_i+2} & \mathbb{Z}_{\nu_i+6} \\ \mathbb{Z}_{\beta_i+i+2} & \mathbb{Z}_{\beta_i+i+3} & \mathbb{Z}_{\beta_i+i+4} & \ldots & \mathbb{Z}_{\beta_i+2i+2} & \mathbb{Z}_{\beta_i+2i+3} & \mathbb{Z}_{\nu_i+7} & \mathbb{Z}_{\nu_i+8} & \mathbb{Z}_{\nu_i+12} \\ \mathbb{Z}_{\beta_i+2i+4} & \mathbb{Z}_{\beta_i+2i+5} & \mathbb{Z}_{\beta_i+2i+6} & \ldots & \mathbb{Z}_{\beta_i+3i+4} & 0 & \mathbb{Z}_{\nu_i+13} & \mathbb{Z}_{\nu_i+14} & \mathbb{Z}_{\nu_i+18} \end{bmatrix}^T \begin{bmatrix} F_x^{*i} \\ F_y^{*i} \\ F_z^{*i} \end{bmatrix},$$

The link weight and the input torque contribute to the generalized active forces by:

$$[G_1^{*i} \quad G_2^{*i} \quad G_5^{*i} \quad \ldots \quad G_{i+3}^{*i} \quad G_{i+4}^{*i} \quad G_{m+5}^{*i} \quad G_{m+6}^{*i} \quad G_{m+7}^{*i}]^T$$
$$= m^i g \begin{bmatrix} \mathbb{Z}_{\beta_i} & \mathbb{Z}_{\beta_i+1} & \mathbb{Z}_{\beta_i+2} & \ldots & \mathbb{Z}_{\beta_i} & \mathbb{Z}_{\beta_i+1} & \mathbb{Z}_{\nu_i+1} & \mathbb{Z}_{\nu_i+2} & \mathbb{Z}_{\nu_i+6} \\ \mathbb{Z}_{\beta_i+i+2} & \mathbb{Z}_{\beta_i+i+3} & \mathbb{Z}_{\beta_i+i+4} & \ldots & \mathbb{Z}_{\beta_i+2i+2} & \mathbb{Z}_{\beta_i+2i+3} & \mathbb{Z}_{\nu_i+7} & \mathbb{Z}_{\nu_i+8} & \mathbb{Z}_{\nu_i+12} \\ \mathbb{Z}_{\beta_i+2i+4} & \mathbb{Z}_{\beta_i+2i+5} & \mathbb{Z}_{\beta_i+2i+6} & \ldots & \mathbb{Z}_{\beta_i+3i+4} & 0 & \mathbb{Z}_{\nu_i+13} & \mathbb{Z}_{\nu_i+14} & \mathbb{Z}_{\nu_i+18} \end{bmatrix}^T \begin{bmatrix} \pi_{z^I}^{x_I^i} \\ \pi_{z^I}^{y_I^i} \\ \pi_{z^I}^{z_I^i} \end{bmatrix},$$
$$T_{i+4}^i = \tau_i$$

3.6. End effector

The end effector is analyzed only as being the point where the load is applied. It makes no inertial contribution by itself; that contribution has been included through the analysis of the preceding links.

Thc end effector, *ee*, is considered a part of the last link in the manipulator chain. Therefore, its angular velocity will be ω^m, since the point belongs to the rigid body m.

If an external load is applied at the end effector of the manipulator, then this can be represented by a couple torque and a force

$$\mathbf{T}_{ee} = T_x^{ee}\mathbf{x}^m + T_y^{ee}\mathbf{y}^m + T_z^{ee}\mathbf{z}^m \qquad \mathbf{F}_{ee} = F_x^{ee}\mathbf{x}^m + F_y^{ee}\mathbf{y}^m + F_z^{ee}\mathbf{z}^m$$

The contribution of this torque and force to the generalized active forces would be

$$[T_1^{ee} \quad T_2^{ee} \quad T_5^{ee} \quad \ldots \quad T_{m+7}^{ee}]^T = \begin{bmatrix} \mathbf{A}_m^T \\ 0 \;\; 0 \;\; 1 \\ \mathbf{M}_m^T \end{bmatrix} \begin{bmatrix} T_x^{ee} \\ T_y^{ee} \\ T_z^{ee} \end{bmatrix},$$

$$[F_1^{ee} \quad F_2^{ee} \quad F_5^{ee} \quad \ldots \quad F_{m+10}^{ee}]^T = \begin{bmatrix} \mathbf{B}_{ee} \\ [\mathbb{X}]_{\pi_{ee}} \\ \mathbf{N}_m^{2^T} \end{bmatrix} \begin{bmatrix} F_x^{ee} \\ F_y^{ee} \\ F_z^{ee} \end{bmatrix},$$

3.7. Dynamic equations

The system dynamic equations can now be derived by adding the contributions of each rigid body

$$\sum (\mathbf{F}^* + \mathbf{F} + \mathbf{T}^* + \mathbf{T}) = 0$$

There will be as many equations as generalized speeds. For generalized speeds u_1, u_2 and u_4:

$$0 = \sum_{j=1}^{w} [T_1^{*w_j} + F_1^{*w_j} + (T_d^{w_j})_1] + T_1^{*v} + F_1^{*v} + T_1^{*0} + F_1^{*0} + F_1^{*1} + T_1^{*1} + G_1^1 + \sum_{i=2}^{m} [T_1^{*i} + F_1^{*i} + G_1^i] + T_1^{ee} + F_1^{ee} \tag{6a}$$

$$0 = \sum_{j=1}^{w} [T_2^{*w_j} + F_2^{*w_j} + (T_d^{w_j})_2] + T_2^{*v} + F_2^{*v} + F_2^{*0} + T_2^{*0} + F_2^{*1} + T_2^{*1} + G_2^1 + \sum_{i=2}^{m} [T_2^{*i} + F_2^{*i} + G_2^i] + T_2^{ee} + F_2^{ee} \tag{6b}$$

$$0 = \sum_{j=1}^{w} \left[\mathbb{Z}_{2_j} \tau_s^{w_j} - \alpha_z^{w_j} \frac{m^{w_j}(r^{w_j})^2}{4} \mathbb{Z}_{2_j} \right] \tag{6c}$$

The dynamic equations that correspond to the manipulator generalized speeds are

$$0=\tau_{r-4}+\sum_{i=r-4}^{m}(T_r^{*i}+F_r^{*i}+G_r^i)+T_r^{ee}+F_r^{ee}, \qquad r\in\{5,\ldots,m+4\} \tag{6d}$$

The dynamic equations that involve the non-contributing forces that have to come into evidence are:

$$0=T_{m+p}^{*0}+T_{m+p}^{0}+G_{m+p}^{1}+T_{m+p}^{*1}+F_{m+p}^{*1}+\sum_{i=2}^{m}(T_{m+p}^{*i}+F_{m+p}^{*i}+G_{m+p}^{i})+T_{m+p}^{ee}+F_{m+p}^{ee},\ p=5,6,7 \tag{6e}$$

$$0=F_{m+s}^{*0}+F_{m+s}^{0}+G_{m+s}^{1}+F_{m+s}^{*1}+\sum_{i=2}^{m}(F_{m+s}^{*i}+G_{m+s}^{i})+F_{m+s}^{ee} \quad s=8,9 \tag{6f}$$

$$0=m^0g+F_{m+10}^{0}+G_{m+10}^{1}+F_{m+10}^{*1}+\sum_{i=2}^{m}(F_{m+10}^{*i}+G_{m+10}^{i})+F_{m+10}^{ee} \tag{6g}$$

In developing the above dynamic equations, care has been taken so that generalized accelerations appear explicitly. This facilitates the transformation of the equations to the closed form:

$$\mathbf{M}\dot{\mathbf{u}}+\mathbf{F}=\mathbf{B}\boldsymbol{\tau} \tag{7}$$

where $\mathbf{M}$ is the inertial matrix, $\dot{\mathbf{u}}$ is the vector of the generalized accelerations, $\mathbf{F}$ is the vector of the Coriolis, centrifugal and gravity terms and $\mathbf{B}$ is the matrix that multiplies the input variable vector $\boldsymbol{\tau}$. In reference [2] explicit formulas for such a transformation are given, although this is actually a matter of algebraic manipulation.

4. CONSTRAINTS

In Section 3, two kinematic constraints have been stated. The rolling without slipping constraint,

$$\mathbf{v}_x^{w_j}=\boldsymbol{\omega}_y^{w_j}\times\mathbf{r}^{w_j}\Rightarrow {}^{w_j}\omega_y^{w_j}=\frac{\dot{x}^v\cos(\phi^{w_j}+\theta)+\dot{y}^v\sin(\phi^{w_j}+\theta)+\dot{\theta}({}^v\ell_x^{w_j}\sin\phi^{w_j}-{}^v\ell_y^{w_j}\cos\phi^{w_j})}{r^{w_j}}$$

and the no skidding constraint for the wheels:

$$v_y^{w_j}=0\Rightarrow\dot{\theta}=\frac{\dot{x}^v\sin(\phi^{w_j}+\theta)-\dot{y}^v\cos(\phi^{w_j}+\theta)}{{}^v\ell_y^{w_j}\sin\phi^{w_j}+{}^v\ell_x^{w_j}\cos\phi^{w_j}} \quad \text{for any } j\in\{1,\ldots,w\}$$

Finally, the relation between the velocities of a wheel center and the vehicle mass center is

$$\mathbf{v}^v=\mathbf{v}^{w_j}-\dot{\theta}\mathbf{z}^v\times\boldsymbol{\ell}^{w_j}$$

The above equation can be used to eliminate another generalized speed, although this option has not been utilized in this paper. The relation can be equivalently described by an equation in the form $\dot{\mathbf{q}}=\mathbf{S}\mathbf{z}$, where $\dot{\mathbf{q}}$ is the vector of the remaining generalized speeds, $\dot{\mathbf{z}}$ is the vector of the independent generalized speeds and $\mathbf{S}$ is a matrix of an appropriate dimension. By differentiating and left-multiplying (7) by $\mathbf{S}^T$, we obtain:

$$\mathbf{S}^T\mathbf{M}\mathbf{S}\dot{\mathbf{z}}+\mathbf{S}^T(\mathbf{M}\dot{\mathbf{S}}\mathbf{z}+\mathbf{F})=\mathbf{S}^T\mathbf{B}\boldsymbol{\tau} \tag{8}$$

Equations (8) are the final, reduced form of the dynamic model of the system, with all constraints included. Nonholonomic constraints have been integrated within the model equations, without the need for Lagrange multipliers. This can ensure that in theory the vehicle trajectory is nonholonomic and that the platform wheels roll without slipping. What ensures, however, is that in real applications the constraints will be respected?

In fact, velocity conditions such as the one used above are only *necessary* and *not sufficient*, because they describe the *result* and *not the cause*. In this section, we treat this problem by completing the set of nonholonomic constraints by additional expressions involving constraint reaction forces at the wheels. We specify the domain of admissible values for the constraint forces and indicate how friction forces will not saturate and how tipping over can be avoided.

The set of constraint equations follows. The no slipping condition is adopted from reference [12] while the tip-over criterion used is the Force-Angle stability measure.[8]

$$\bar{\mathbf{R}}\mathbf{z}^I = \sum_{j=1}^{w} \mathbf{F}^{w_j}\mathbf{z}^I \quad \text{(vertical balance)} \qquad 0 \le F_z^{w_j} = \mathbf{F}^{w_j}\mathbf{z}^{w_j}$$

$$r^{w_j}\mathbf{F}^{w_j}\mathbf{x}^{w_j} = \tau_d^{w_j} \quad \text{(driving torques)} \qquad 0 \ge \frac{1}{\sqrt{1+\eta}}\|\mathbf{F}^{w_j}\| - \mathbf{z}^{w_j}\mathbf{F}^{w_j} \quad \text{(no slipping)} \tag{9}$$

$$\left(\bar{\mathbf{R}} + \sum_{j=1}^{w} \mathbf{F}^{w_j}\mathbf{F}^{w_j}\right) \cdot (\boldsymbol{\omega}^v \times \mathbf{v}^v) = 0 \quad \text{(no skidding)} \qquad \alpha = \min(\vartheta_i)\|\bar{\mathbf{R}}\| > 0 \quad \text{(no tipover)}$$

where ϑ_i is the angle measure associated with each tip-over axis, i, given by $\vartheta_i = sign\{(\hat{\mathbf{l}}_i \times \hat{\mathbf{f}}_i^*) \cdot \hat{\mathbf{a}}_i\}\ \arccos(\hat{\mathbf{f}}_i^* \cdot \hat{\mathbf{l}}_i)$, the reaction forces exerted on the wheels are denoted by $\mathbf{F}^{w_j}$, $\bar{\mathbf{R}}$ is the resultant of all contact and field forces exerted on the platform given by $\bar{\mathbf{R}} = m^v(\mathbf{g} - \mathbf{a}^v) + \sum_{j=1}^{w} m^{w_j}(\mathbf{g} - \mathbf{a}^{w_j}) + \mathbf{F}^{\bar{0}}$, η is the coefficient of static friction between a wheel and the ground, and the rest of the symbols are defined as

$$\mathbf{f}_i^* = \mathbf{f}_i + \frac{\hat{\mathbf{l}}_i \times \mathbf{n}_i}{\|\mathbf{l}_i\|} \qquad \hat{\mathbf{l}}_i = \frac{\mathbf{l}_i}{\|\mathbf{l}_i\|} \qquad \hat{\mathbf{f}}_i^* = \frac{\mathbf{f}_i^*}{\|\mathbf{f}_i^*\|} \qquad \hat{\mathbf{a}}_i = \frac{\mathbf{a}_i}{\|\mathbf{a}_i\|} \qquad \mathbf{a}_i = \boldsymbol{\ell}^{w_{i+1}} - \boldsymbol{\ell}^{w_i}$$

$$\mathrm{l}_i = (\mathbf{I} - \hat{\mathbf{a}}_i\hat{\mathbf{a}}_i^T)({}^v\boldsymbol{\ell}^{w_{i+1}} - r^{w_i}\mathbf{z}^v - {}^v\mathbf{p}^{vcm}) \quad \mathbf{f}_i = (\mathbf{I} - \hat{\mathbf{a}}_i\hat{\mathbf{a}}_i^T)\hat{\mathbf{R}} \quad \mathbf{n}_i = -\sum_{j=1}^{w} \mathbf{T}^{*w_j} - \mathbf{T}^{*v} - \mathbf{T}^{*0} + \mathbf{T}^{\bar{0}}$$

Equations (6) along with constraints (9) constitute a dynamic model of a mobile manipulator under load with the full set of constraints imposed on the system.

5. DISCUSSION

5.1. Physical insight

Physical insight stems from the fact that in the final expressions, one is capable of distinguishing the forces and torques that influence the dynamic behavior of the mechanism. Take, for example, (6), in the case of a mobile manipulator with two horizontal links and no load (see Section 6):

$$0 = \sum_{j=1}^{w} [T_1^{*w_j} + F_1^{*w_j} + (T_d^{w_j})_1] + T_1^{*v} + F_1^{*v} + T_1^{*0} + F_1^{*0} + F_1^{*1} + T_1^{*1} + T_1^{*2} + F_1^{*2}$$

$$0 = \sum_{j=1}^{w} [T_2^{*w_j} + F_2^{*w_j} + (T_d^{w_j})_2] + T_2^{*v} + F_2^{*v} + T_2^{*0} + F_2^{*0} + F_2^{*1} + T_2^{*1} + T_2^{*2} + F_2^{*2}$$

$$0 = \sum_{j=1}^{w} \left[\mathbb{Z}_{2_j}\tau_s^{w_j} - \alpha_z^{w_j}\frac{m^{w_j}(r^{w_j})^2}{4}\mathbb{Z}_{2_j}\right]$$

$$0 = \tau_1 + T_5^{*1} + F_5^{*1} + T_5^{*2} + F_5^{*2}$$

$$0 = \tau_2 + T_6^{*1} + F_6^{*1} + T_6^{*2} + F_6^{*2}$$

It is evident that the only contributing forces are the inertial forces/torques and the input torques. The weight does not contribute because the motion is planar. Nor do any nonholonomic constraint forces and therefore they do not have to be calculated. On this issue, the approach departs from conventional methodologies which require calculation or elimination of the constraint forces.

By examining the system from Kane's methodology viewpoint, it is evident that when having the complete model of the mobile manipulator the interaction forces between the manipulator and the mobile platform do not contribute to dynamics and therefore they have no impact on the system's performance. Therefore, contrary to existing approaches, the control design need not focus on cancellation of this interaction. Such control approaches[5] could be justified within a framework of decentralized control, but cannot promise improved dynamic performance.

What is also clear is the effect of ignoring the friction at the wheels: an infinitesimal steering torque will cause an undamped rotation of the wheels.

The most important characteristic of the nonholonomic attribute of the system, reflected on the dynamic equations is with no doubt the lack of correspondence between the number of degrees of freedom and the number of dynamic equations.

5.2. Computational complexity

Computational complexity is another important issue. Kane and Levinson[4] modeled a Stanford Manipulator and reported fewer multiplications and additions than any other approach. Kane, however has not described analytically the complexity of his computational algorithm. We have derived an upper bound for the calculations our model requires:

$$\text{multiplications: } 37.5m^2+441.5m+88+28w, \quad \text{additions: } 31.5m^2+392.5m+26+11w,$$

where m is the number of manipulator links and w is the number of the vehicle wheels. Both the number of multiplications and additions belong to $O(m^2)$.

5.3. Comparison to alternative methodologies

The algorithm is obviously superior to Lagrange formulations as far as computational complexity is concerned, since the Lagrange computational algorithm is in $O(m^4)$.

As far as Newton-Euler formulation is concerned, it must be noted that the parabolic curves presented above lie below the Newton's line for reasonable numbers of system degrees of freedom. To make a simple comparison, consider the case of a vehicle with four wheels one manipulator link. The number of rigid bodies in the system is 6. Our algorithm will require in the worst case,

Kane:	679 multiplications	494 additions

To estimate the number of calculations needed for a Newton-Euler algorithm, we borrow the polynomials derived for the case of a Puma manipulator with 6 links[13] which gives:

Newton:	678 multiplications	597 additions

Moreover, Newton-Euler formulation exhibits difficulties in dealing with reaction forces at the wheels. Its recursive nature requires the calculations of all forces and torques exerted on the system. However, the distribution of the reaction forces to the wheels is unknown.[14] Finally, it must be emphasized that our equations are not optimized in terms of complexity, and are presented in the most general form. Care has been taken so that generalized accelerations appear explicitly and the equations can easily be brought into the form $M\ddot{x}+C(x)\dot{x}+K(x)=F$, which is suitable for control. The same cannot be said, however, for Newton-Euler equations.

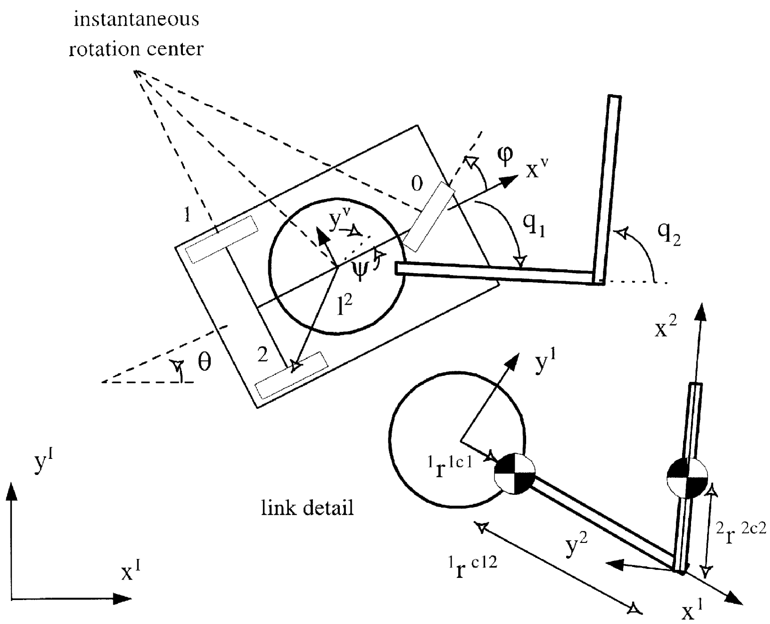

Fig. 2. Mobile manipulator used in the example.

6. NUMERICAL VERIFICATION

This example is a three wheel, two link mobile manipulator (Figure 2). The geometric and dynamic parameters used are given in Table I.

Simulations were conducted in *MATLAB* using an *ode15s* integrator. Input torques were set to zero. For a number of motion cases, Figures 3–5 depict the motion of the mobile manipulator, the total energy of the system, the friction forces and the influence of forces and torques on stability against tip-over.

Monitoring the total energy of the system can help verify the validity since in the absence of input torques, the Hamiltonian of the system should remain invariant. In the planar examples, this Hamiltonian reduces to the kinetic energy of the system. As can be seen in all simulations, the kinetic energy of the mechanism remains constant, implying that the system model is consistent.

The calculation of friction forces serves in determining stability margins against slipping and skidding. Friction forces are directly related to velocities and accelerations, as shown in Section 4. Ensuring small values for the necessary centripetal force can safeguard against the risk of skidding. Modeling the characteristics of tire friction is quite demanding and is still an issue of research in progress. Friction models[10,15] can serve towards this direction and significant effort is currently devoted to extending their applicability and generality.

Another important measure for mechanical stability is the tip-over stability measure. Normally this measure should take positive values. Negative values indicate that the system is on the verge of tipping over, having lost contact of some of its wheels with the ground. Whether tip-over will occur depends on the time interval for which the measure remains negative and the configuration of the system in

Table I. Geometric and dynamic parameters used in the example system.

Geometric Parameters (*m*)

$l_x^{w_0}=0.3$	$l_y^{w_0}=0$	$l_y^{w_2}=-0.3$	${}^v r_y^0=0$	${}^v r_y^{01}=0$	${}^v r_x^{01}=0$	${}^1 r_x^{1c_1}=0.25$
$l_y^{w_1}=0.3$	$l_x^{w_1}=-0.3$	$l_x^{w_2}=-0.3$	${}^v r_x^0=0$	${}^v r_z^{01}=0$	${}^1 r_y^{1c_1}=0$	${}^1 r_z^{1c_1}=0.1505$
${}^1 r_x^{c_1 2}=0.8$	${}^1 r_y^{c_1 2}=0$	${}^1 r_z^{c_1 2}=0$	$r^w=0.15$	${}^2 r_y^{2c_2}=0$	${}^2 r_z^{2c_2}=0$	${}^2 r_x^{2c_2}=0.4$

Inertial Parameters (*kg* or $kg \cdot m^2$)

$m^w=2$	$m^1=10$	$m^2=5$	$I_x^v=1.667$	$I_z^v=3.02$	$I_y^0=0.08$	$I_y^1=2.035$	$I_x^1=1.759$	$I_z^2=0.267$
$m^0=5$	$m^v=40$		$I_y^v=1.667$	$I_x^0=0.08$	$I_z^0=0.156$	$I_z^1=2.073$	$I_y^2=0.267$	$I_x^2=0$

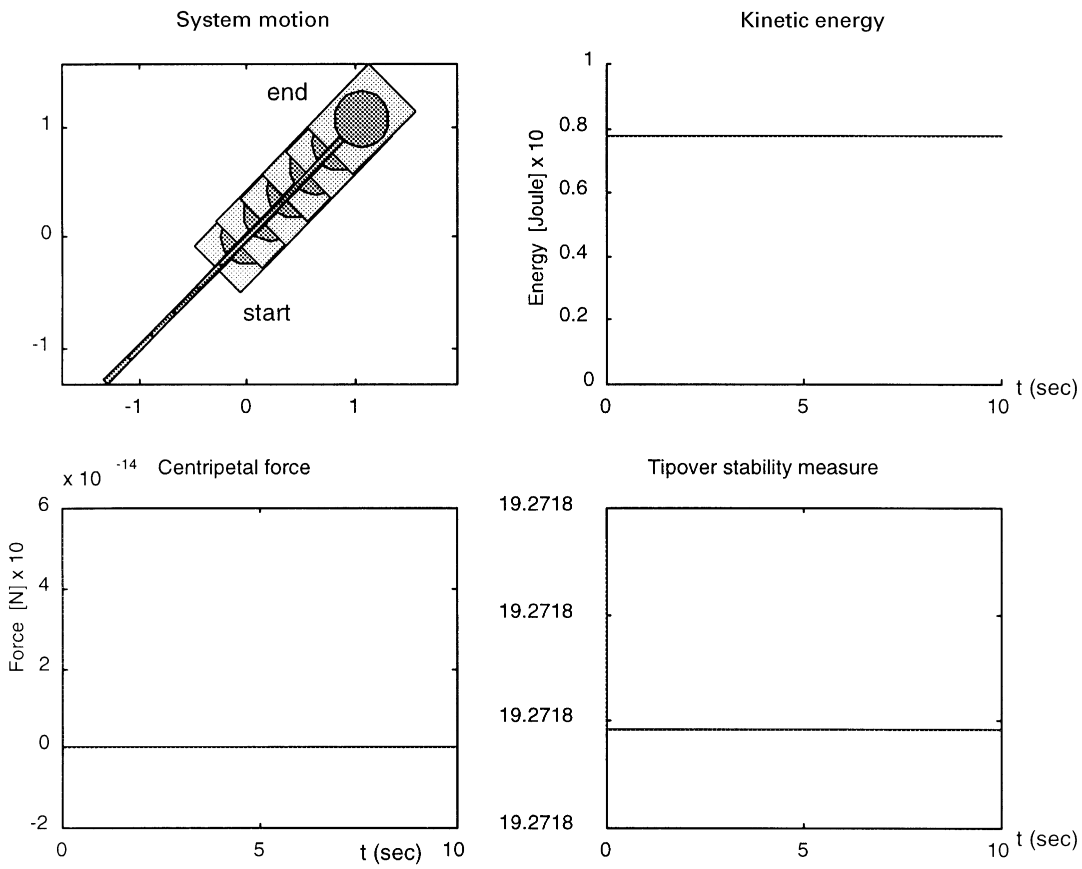

Fig. 3. Motion in straight line with aligned links.

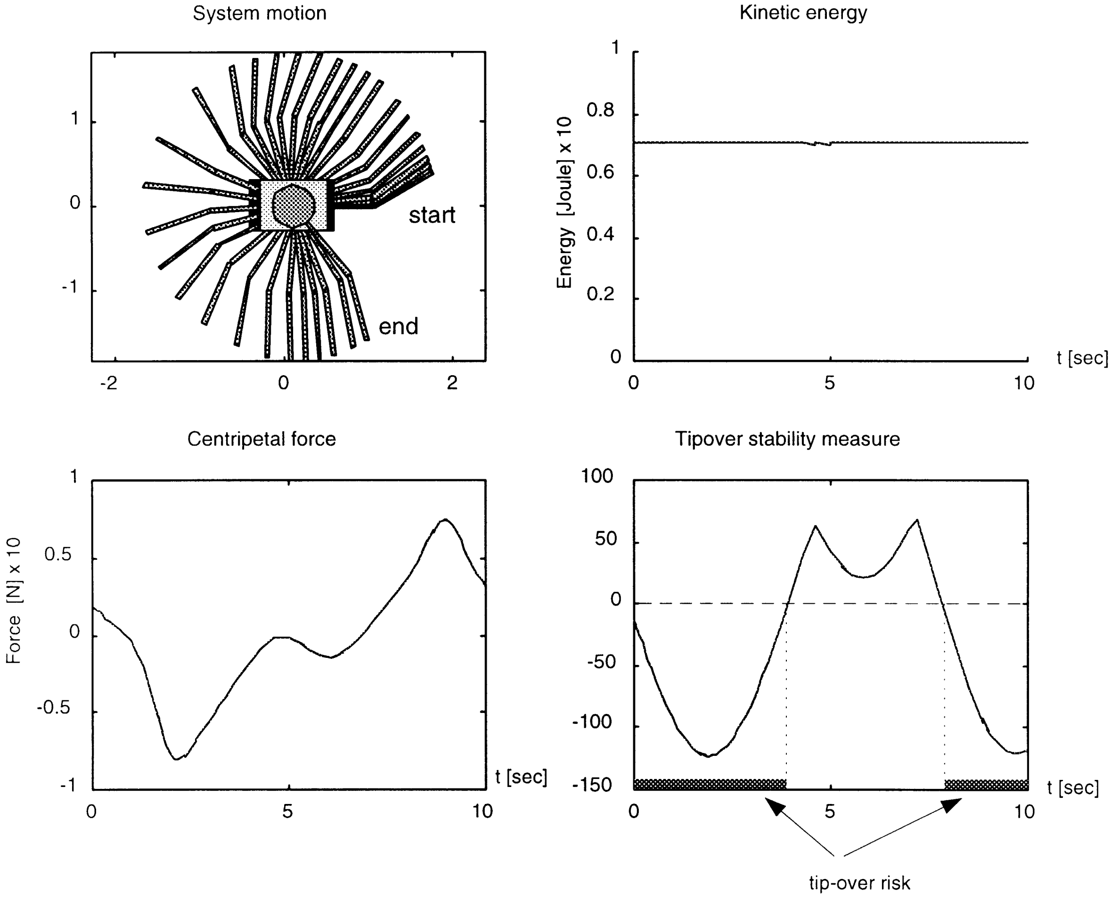

Fig. 4. Circular motion of the first link.

terms of the ability of gravitational stabilizing forces a moment at that particular configuration to bring the mechanism back in full contact with the ground.[8] In real applications, this quantity should always remain positive. In simulations however we have the advantage to let this measure vary freely in order to determine the kind of motion that is most likely to cause tip-over. In this perspective, in the type of motion depicted in Figures 4 and 5 the risk of tip-over is substantial.

In Figure 3 the mobile platform has an initial translational velocity and moves in a straight line while the two links are aligned with the direction of motion. As expected, due to the straight line motion of all components, no centripetal forces are developed. The tip-over stability measure is positive indicating no risk of tip-over. Since no input torques are applied, the kinetic energy remains constant at all times and the motion produced in the simulations verifies that the robot maintains its straight motion.

In Figure 4 the base is stationary and the first link rotates while the second joint is unactuated (passive). This motion generates inertial forces which enforce nonholonomic motion on the base. The nonholonomic motion is depicted in Figure 4 in the horizontal oscillation of the platform indicated by the shaded region. The dynamic effect of the rotation of the passive link is depicted in the varying velocity of the first link (successive plots correspond to constant time steps). Centripetal forces accelerate the links and produce the irregular spacing between each link configuration. This motion can cause tip-over during the time periods indicated in Figure 4 by the thick lines on the time axis of the graph representing the tip-over stability measure. During the time when the first link angle is close to $\pi/2$ and $3\pi/2$ the centrifugal forces are sufficient to cause tip-over.

Finally, in Figure 5, the base moves on a circular path while the manipulator links are unactuated. This case is useful to observe how the motion of the links affects the mobile platform's speed and how the centrifugal forces induced by the circular motion of the platform cause link rotation. Thc influence on platform speed is evident from the irregular spacing between successive platform locations. Tip-over can occur during the time periods indicated by the thick lines on the time axis of the tip-over stability measure graph, corresponding to the configurations where the first link is ovcr the right side of the mobile platform.

7. CONCLUSIONS AND ISSUES FOR FURTHER RESEARCH

In this paper a complete model for a mobile manipulator system has been constructed, consisted of a set of dynamic equations and a set of constraint equations. The model is formulated following Kane's approach which is advantageous to conventional approaches. Within that framework, the importance of the dynamic interaction between the vehicle and the attached manipulator for stability was emphasized and the methods for bringing such an interaction into evidence were exploited. Monitoring the reaction forces from the ground is necessary in order to secure the

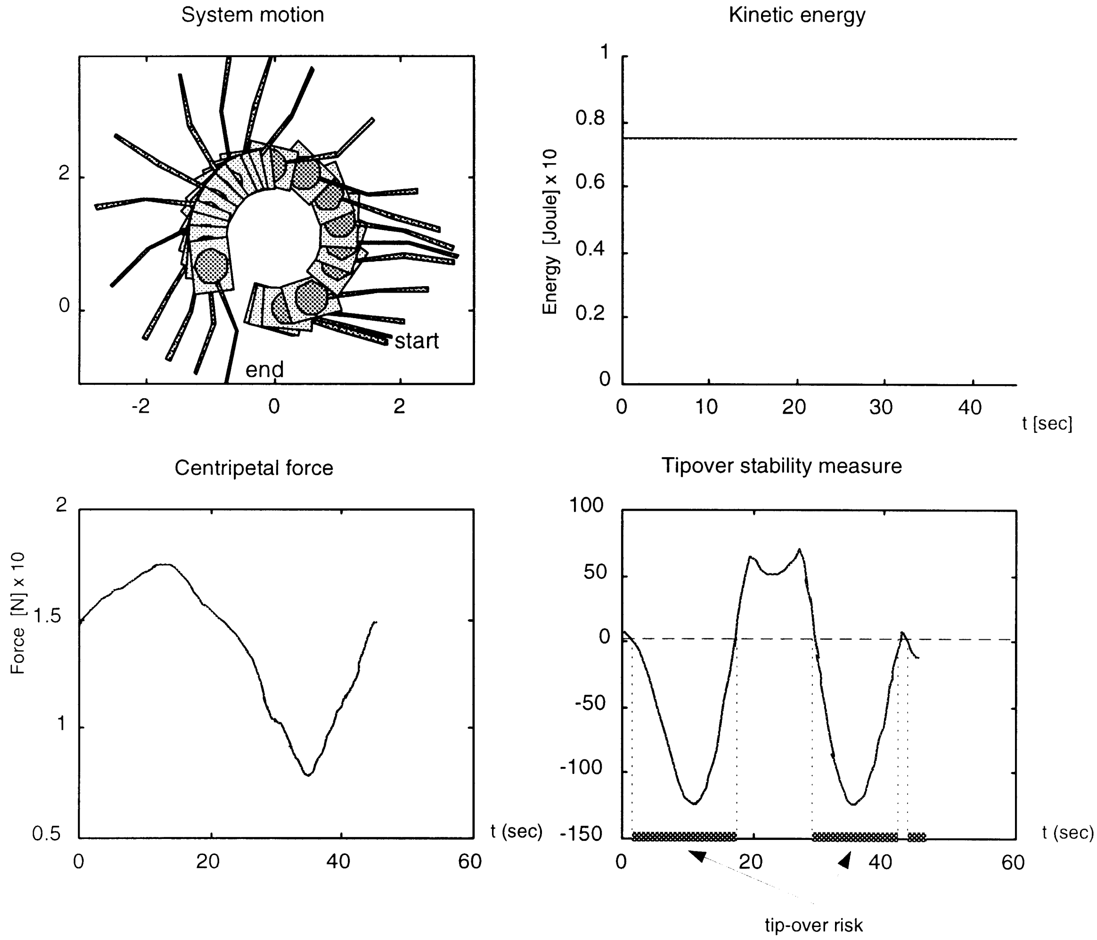

Fig. 5. Circular motion of the platform with unactuated arm.

system from tipping over and to guarantee that nonholonomic constraints are respected. It is pointed out that conventional velocity nonholonomic constraint equations are not sufficient to impose a nonholonomic motion unless accompanied with their dynamic counterparts, which are formally expressed and presented in this paper. Current issues of interest include modeling multiple cooperating mobile manipulator systems to enable effective control strategies.

References

1. G. Campion, B. d'Andrea Novel, and G. Bastin, "Modelling and state feedback control of nonholonomic mechanical systems", *Proceedings of the 1991 IEEE Conference on Decision and Control* (December, 1991).
2. T. J. Tarn, G. A. Shoults, and S. P. Yang, "Dynamic model for an underwater vehicle with multiple robotic manipulators", *Pre-Proc. 6th Int. Advanced Robotics Program,* Toulon-La Seyne (1996) pp. 1–23.
3. O. Khatib. K. Yokoi, D. Ruspini, R. Holmberg, A. Casal, and A. Baader, "Force strategies for cooperative tasks in multiple mobile manipulation systems", *Int. J. Robotics Research* **12**(7), 333–342 (1996).
4. Thomas R. Kane and David A. Levinson, "The use of Kane's dynamical equations in robotics", *Int. J. Robotics Research* **2**(3), 3–21 (October, 1996).
5. Yoshio Yamamoto, "Control and Coordination of Locomotion and Manipulation of a Wheeled Mobile Manipulator", *PhD thesis* (Dept. of Computer and Information Science, School of Engineering and Applied Science, University of Pennsylvania, Philadelphia, PA, August 1994).
6. M. W. Chen and A. M. S. Zalzala, "Dynamic modelling and genetic-based motion planning of mobile manipulator systems with nonholonomic constraints", *Research Report 600* (Robotics Research Group, Dept. of Automatic Control and Systems Engineering, University of Sheffield, UK, September 1995).
7. K. Thanjavur and R. Rajogopalan "Ease of dynamic modelling of wheeled mobile robots (WMRs) using Kane's approach", *IEEE Int. Conf. Rob. and Autom.*, Albuquerque, New Mexico (1997) pp. 2926–2931.
8. E. G. Papadopoulos and D. A. Rey, "A new measure of tipover stability margin for mobile manipulators", *IEEE Int. Conf. Robotics and Automation*, Minneapolis, MN (April, 1996).
9. H. G. Tanner and K. J. Kyriakopoulos, "Modeling of a mobile manipulator with the use of Kane's dynamical equations", *IFAC Meeting on Intelligent Autonomous Vehicles,* Madrid (March, 1998) pp. 14–40.
10. C. Canudas de Wit and P. Tsiotras, "Dynamic tire friction models for vehicle traction control", *Proc. of the 1999 IEEE International Conference on Decision and Control*, Phoenix, Arizona (December, 1999) pp. 3746–3751.
11. R. Thomas Kane and A. David Levinson, *Dynamics: Theory and Applications*(McGraw-Hill, 1985).
12. Yoshihiko Nakamura, *Advanced Robotics: Redundancy and Optimization* (Addison-Wesley, 1991).
13. K. S. Fu, R. C. Gonzalez, and C. S. G. Lee, *Robotics. Control, Sensing, Vision, and Intelligence* (McGraw-Hill, 1989).
14. S. Dubowsky and E. E. Vance, "Planning mobile manipulator motions considering vehicle dynamic stability constraints", *Proc. IEEE Int. Conf. on Robotics and Automation* (1989) pp. 1271–1276.
15. H. Olson, K. J. Aström, C. Canudas de Wit, M. Gäfrert, and P. Lischinsky, "Friction models and friction compensation", *European Journal of Control* **4**, 176–195 (December, 1998).

APPENDIX

Let m be the total number of manipulator links and $i>1$

$$\lambda=3+9m+\frac{3(m-1)m}{2},\qquad \kappa=3m^2+17m+11,\qquad \xi=3m^2+20m+14,$$

$$\mu=3m^2+23m+18,\qquad \nu=3m^2+32m+18,\qquad \alpha_i=4+9(i-1)+\frac{3(i-2)(i-1)}{2},$$

$$\delta_i=3m^2+20m+15+3i,\qquad \varepsilon_i=5i-6+\frac{3(i-2)(i-1)}{2},\qquad \mu_i=3m^2+23m+9+9i,$$

$$\beta_i=12+9m+\frac{3(m-1)m}{2}+11(i-1)+\frac{3(i-2)(i-1)}{2},\qquad \gamma_i=3m^2+17m+11+3i,$$

$$\nu_i=3m^2+32m+18i,\qquad \pi_i=\frac{3(m-2)(m-1)}{2}+8m-22+9i,$$

$$c\theta=\cos\ \theta,\qquad s\theta=\sin\ \theta,\qquad c\phi_{w_j}=\cos\ \phi^{w_j},\qquad \mathbb{Z}_o={}^{v}\ell_y^{w_o}s\phi_o+{}^{v}\ell_x^{w_o}c\phi_o,$$

$$c\phi_o=\cos\ \phi^{o},\qquad s\phi_o=\sin\ \phi^{o},\qquad c\phi\theta_{w_j}=\cos(\phi^{w_j}+\theta),\qquad \mathbb{Z}_{3_j}=\mathbb{Z}_o\,c\phi\theta_{w_j}-\mathbb{Z}_{1_j}\mathbb{Z}_1,$$

$$s\phi_{w_j}=\sin\ \phi^{w_j},\qquad \mathbb{Z}_9=\pi_{z^1}^{z^v}\mathbb{Z}_2,\qquad s\phi\theta_{w_j}=\sin(\phi^{w_j}+\theta),\qquad \mathbb{Z}_{1_j}={}^{v}\ell_y^{w_j}c\phi_{w_j}-{}^{v}\ell_x^{w_j}s\phi_{w_j},$$

$$\mathbb{Z}_5=\pi_{x^1}^{z^v}\mathbb{Z}_2,\qquad \mathbb{Z}_2=\cos(\phi_o+\theta),\qquad \mathbb{Z}_1=\sin(\phi_o+\theta),\qquad \mathbb{Z}_{4_j}=\mathbb{Z}_o s\phi\theta_{w_j}-\mathbb{Z}_{1_j}\mathbb{Z}_2,$$

$$\mathbb{Z}_{2_j}=\frac{\partial\phi^{w_j}}{\partial\phi},\qquad \mathbb{Z}_3=\mathbb{Z}_1u_1+\mathbb{Z}_2u_2,\qquad \mathbb{Z}_4=\pi_{x^1}^{z^v}\mathbb{Z}_1,\qquad \mathbb{Z}_{7_j}={}^{v}\ell_y^{w_j}s\phi_{w_j}+{}^{v}\ell_x^{w_j}c\phi_{w_j},$$

$$\mathbb{Z}_6=\pi_{y^1}^{z^v}\mathbb{Z}_1,\qquad \mathbb{Z}_{10}=\mathbb{Z}_4u_1+\mathbb{Z}_5u_2\qquad \mathbb{Z}_8=\pi_{z^1}^{z^v}\mathbb{Z}_1,\qquad \mathbb{Z}_{8_o}={}^{v}\ell_y^{w_o}c\phi_o-{}^{v}\ell_x^{w_o}s\phi_o,$$

$$\mathbb{Z}_7=\pi_{y^1}^{z^v}\mathbb{Z}_2,\qquad \mathbb{Z}_{11}=\mathbb{Z}_6u_1+\mathbb{Z}_7u_2\qquad \mathbb{Z}_{12}=\mathbb{Z}_8u_1+\mathbb{Z}_9u_2+u_5$$

$$\mathbb{Z}_{\lambda+1}=\mathbb{Z}_o u_1,\qquad \mathbb{Z}_{\lambda+9}=\pi_{x^1}^{x^v}\mathbb{X}_1+\pi_{x^1}^{y^v}\mathbb{X}_2+{}^{1}r_z^{1c_1}\mathbb{Z}_6-{}^{1}r_y^{1c_1}\mathbb{Z}_{78},\qquad \mathbb{Z}_{\lambda+2}=\mathbb{Z}_o u_2,$$

$$\mathbb{Z}_{\lambda+4}=\mathbb{Z}_o s\theta-{}^{v}r_y^{0}\mathbb{Z}_2,\qquad \mathbb{Z}_{\lambda+10}=\pi_{x^1}^{x^v}\mathbb{X}_3+\pi_{x^1}^{y^v}\mathbb{X}_4+{}^{1}r_z^{1c_1}\mathbb{Z}_7-{}^{1}r_y^{1c_1}\mathbb{Z}_9,\qquad \mathbb{Z}_{\lambda+5}={}^{v}r_x^{0}\mathbb{Z}_1-\mathbb{Z}_o s\theta,$$

$$\mathbb{Z}_{\lambda+3}=\mathbb{Z}_o c\theta-{}^{v}r_y^{0}\mathbb{Z}_1,\qquad \mathbb{Z}_{\lambda+12}=\pi_{y^1}^{x^v}\mathbb{X}_1+\pi_{y^1}^{y^v}\mathbb{X}_2+{}^{1}r_z^{1c_1}\mathbb{Z}_8-{}^{1}r_2^{1c_1}\mathbb{Z}_4,\qquad \mathbb{Z}_{\lambda+6}={}^{c}r_x^{0}\mathbb{Z}_2+\mathbb{Z}_o c\theta,$$

$$\mathbb{Z}_{\lambda+7}=\mathbb{Z}_{\lambda+3}u_1+\mathbb{Z}_{\lambda+4}u_2,\qquad \mathbb{Z}_{\lambda+13}=\pi_{y^1}^{x^v}\mathbb{X}_3+\pi_{y^1}^{y^v}\mathbb{X}_4+{}^{1}r_x^{1c_1}\mathbb{Z}_9-{}^{1}r_z^{1c_1}\mathbb{Z}_5,\qquad \mathbb{Z}_{\lambda+11}=-{}^{1}r_y^{1c_1},$$

$$\mathbb{Z}_{\lambda+8}=\mathbb{Z}_{\lambda+5}u_1+\mathbb{Z}_{\lambda+6}u_2,\qquad \mathbb{Z}_{\lambda+15}=\pi_{z^1}^{x^v}\mathbb{X}_1+\pi_{z^1}^{y^v}\mathbb{X}_2+{}^{1}r_y^{1c_1}\mathbb{Z}_4-{}^{1}r_x^{1c_1}\mathbb{Z}_6,\qquad \mathbb{Z}_{\lambda+14}={}^{1}r_x^{1c_1},$$

$$\mathbb{Z}_{\lambda+19}=\mathbb{Z}_{\lambda+15}u_1+\mathbb{Z}_{\lambda+16}u_2,\qquad \mathbb{Z}_{\lambda+16}=\pi_{z^1}^{x^v}\mathbb{X}_3+\pi_{z^1}^{y^v}\mathbb{X}_4+{}^{1}r_y^{1c_1}\mathbb{Z}_5-{}^{1}r_x^{1c_1}\mathbb{Z}_7,$$

$$\mathbb{Z}_{\lambda+18}=\mathbb{Z}_{\lambda+12}u_1+\mathbb{Z}_{\lambda+13}u_2+\mathbb{Z}_{\lambda+14}u_5,\qquad \mathbb{Z}_{\lambda+17}=\mathbb{Z}_{\lambda+9}u_1+\mathbb{Z}_{\lambda+10}u_2+\mathbb{Z}_{\lambda+11}u_5,$$

$$\mathbb{X}_1=\mathbb{Z}_{\lambda+3}-{}^{v}r_y^{01}\mathbb{Z}_1,\qquad \mathbb{X}_2=\mathbb{Z}_{\lambda+5}+{}^{v}r_x^{01}\mathbb{Z}_1,\qquad \mathbb{X}_3=\mathbb{Z}_{\lambda+4}-{}^{v}r_y^{01}\mathbb{Z}_2,$$

$$\mathbb{X}_4=\mathbb{Z}_{\lambda+6}+{}^{v}r_x^{01}\mathbb{Z}_2,\qquad \mathbb{X}_5={}^{1}r_z^{c_12}\mathbb{Z}_6-{}^{1}r_y^{c_12}\mathbb{Z}_8+\mathbb{Z}_{\lambda+9},\qquad \mathbb{X}_6={}^{1}r_x^{c_12}\mathbb{Z}_8-{}^{1}r_z^{c_12}\mathbb{Z}_4+\mathbb{Z}_{\lambda+12},$$

$$\mathbb{X}_7={}^{1}r_y^{c_12}\mathbb{Z}_4-{}^{1}r_x^{c_12}\mathbb{Z}_6+\mathbb{Z}_{\lambda+15},\qquad \mathbb{X}_8={}^{1}r_z^{c_12}\mathbb{Z}_7-{}^{1}r_y^{c_12}\mathbb{Z}_9+\mathbb{Z}_{\lambda+10},\qquad \mathbb{X}_9={}^{1}r_x^{c_12}\mathbb{Z}_9-{}^{1}r_z^{c_12}\mathbb{Z}_5+\mathbb{Z}_{\lambda+13},$$

$$\mathbb{X}_{10}={}^{1}r_y^{c_12}\mathbb{Z}_5-{}^{1}r_x^{c_12}\mathbb{Z}_7+\mathbb{Z}_{\lambda+16},\qquad \mathbb{X}_{11}=-{}^{1}r_y^{c_12}+\mathbb{Z}_{\mathbf{l}+11},\qquad \mathbb{X}_{12}={}^{1}r_x^{c_12}+\mathbb{Z}_{\lambda+14},$$

$$\mathbb{Z}_{\kappa+1}=(\mathbb{Z}_1 u_2-\mathbb{Z}_2 u_1)\left(\frac{\partial\phi^o}{\partial\phi}+Z_3\right)u_4, \qquad \mathbb{Z}_{(\kappa+2)_j}=\left\{\mathbb{Z}_{8o}[u_1 c\phi\theta_{w_j}+u_2 s\phi\theta_{w_j}]\frac{\partial\phi^o}{\partial\phi}-\mathbb{Z}_{7_j}\mathbb{Z}_{2_j}\mathbb{Z}_3\right\}u_4-\mathbb{Z}_{1_j}\mathbb{Z}_{\kappa+1}$$

$$+\mathbb{Z}_o(\mathbb{Z}_{2_j}u_4+\mathbb{Z}_3)(u_2 c\phi\theta_{w_j}-u_1 s\phi\theta_{w_j}),$$

$$\mathbb{Z}_{(\kappa+3)_j}=\mathbb{Z}_{\kappa+1}+\frac{\partial^2\phi^{w_j}}{\partial\phi^2}u_4^2, \qquad \mathbb{Z}_{\kappa+4}=(\pi^{z^v}_{x^1})_{q_1}\mathbb{Z}_3 u_5+\pi^{z^v}_{x^1}\mathbb{Z}_{\kappa+1},$$

$$\mathbb{Z}_{\kappa+5}=(\pi^{z^v}_{y^1})_{q_1}\mathbb{Z}_3 u_5+\pi^{z^v}_{y^1}\mathbb{Z}_{\kappa+1}, \qquad \mathbb{Z}_{\kappa+6}=(\pi^{z^v}_{z^1})_{q_1}\mathbb{Z}_3 u_5+\pi^{z^v}_{z^1}\mathbb{Z}_{\kappa+1},$$

$$\mathbb{Z}_{\xi+1}=\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4 u_1-\mathbb{Z}_3\mathbb{Z}_o u_2, \qquad \mathbb{Z}_{\xi+3}=\mathbb{Z}_{8o}\mathbb{Y}_1\frac{\partial\phi^o}{\partial\phi}u_4-{}^v r_y^0\mathbb{Z}_{\kappa+1}+\mathbb{Z}_3(\mathbb{Z}_o\mathbb{Y}_2-\mathbb{Z}_{\lambda+8}),$$

$$\mathbb{Z}_{\xi+2}=\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4 u_2+\mathbb{Z}_3\mathbb{Z}_o u_1, \qquad \mathbb{Z}_{\xi+4}=\mathbb{Z}_{8o}\mathbb{Y}_2\frac{\partial\phi^o}{\partial\phi}u_4+{}^v r_x^0\mathbb{Z}_{\kappa+1}-\mathbb{Z}_3(\mathbb{Z}_o\mathbb{Y}_1-\mathbb{Z}_{\lambda+7}),$$

$$\mathbb{Z}_{\xi+5}=\mathbb{Y}_3+\mathbb{Z}_{\lambda+19}\mathbb{Z}_{11}-\mathbb{Z}_{\lambda+18}\mathbb{Z}_{12}, \qquad \mathbb{Z}_{\xi+6}=\mathbb{Y}_4+\mathbb{Z}_{\lambda+17}\mathbb{Z}_{12}-\mathbb{Z}_{\lambda+19}\mathbb{Z}_{10},$$

$$\mathbb{Z}_{\xi+7}=\mathbb{Y}_5+\mathbb{Z}_{\lambda+18}\mathbb{Z}_{11}-\mathbb{Z}_{\lambda+17}\mathbb{Z}_{11},$$

$$\mathbb{Y}_1=c\theta u_1+s\theta u_2, \qquad \mathbb{Y}_2=c\theta u_2-s\theta u_1,$$

$$\mathbb{Y}_3=\{[(\pi^{x^v}_{x^1})_{q_1}\mathbb{X}_1+(\pi^{y^v}_{x^1})_{q_1}\mathbb{X}_2]u_1+[(\pi^{x^v}_{x^1})_{q_1}\mathbb{X}_3+(\pi^{y^v}_{x^1})_{q_1}\mathbb{X}_4]u_2\}u_5+\pi^{x^v}_{x^1}\left(\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4\mathbb{Y}_1+\mathbb{Z}_o\mathbb{Z}_3\mathbb{Y}_2\right)$$

$$+\pi^{y^v}_{x^1}\left(\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4\mathbb{Y}_2-\mathbb{Z}_o\mathbb{Z}_3\mathbb{Y}_1\right)+[({}^v r_x^0+{}^v r_x^{01})\pi^{y^v}_{x^1}-({}^v r_y^0+{}^v r_y^{01})\pi^{x^v}_{x^1}]\mathbb{Z}_{\kappa+1}+{}^1 r_z^{1c_1}\mathbb{Z}_{\kappa+5}-{}^1 r_y^{1c_1}\mathbb{Z}_{\kappa+6},$$

$$\mathbb{Y}_4=\{[(\pi^{x^v}_{y^1})_{q_1}\mathbb{X}_1+(\pi^{y^v}_{y^1})_{q_1}\mathbb{X}_2]u_1+[(\pi^{x^v}_{y^1})_{q_1}\mathbb{X}_3+(\pi^{y^v}_{y^1})_{q_1}\mathbb{X}_4]u_2\}u_5+\pi^{x^v}_{y^1}\left(\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4\mathbb{Y}_1+\mathbb{Z}_o\mathbb{Z}_3\mathbb{Z}_2\right)$$

$$+\pi^{y^v}_{y^1}\left(\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4\mathbb{Y}_2-\mathbb{Z}_o\mathbb{Z}_3\mathbb{Y}_1\right)+[({}^v r_x^0+{}^v r_x^{01})\pi^{y^v}_{y^1}-({}^v r_y^0+{}^v r_y^{01})\pi^{x^v}_{y^1}]\mathbb{Z}_{\kappa+1}+{}^1 r_z^{1c_1}\mathbb{Z}_{\kappa+6}-{}^1 r_y^{1c_1}\mathbb{Z}_{\kappa+4},$$

$$\mathbb{Y}_5=\{[(\pi^{x^v}_{z^1})_{q_1}\mathbb{X}_1+(\pi^{y^v}_{z^1})_{q_1}\mathbb{X}_2]u_1+[(\pi^{x^v}_{z^1})_{q_1}\mathbb{X}_3+(\pi^{y^v}_{z^1})_{q_1}\mathbb{X}_4]u_2\}u_5+\pi^{x^v}_{z^1}\left(\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4\mathbb{Y}_1+\mathbb{Z}_o\mathbb{Z}_3\mathbb{Y}_2\right)$$

$$+\pi^{y^v}_{z^1}\left(\mathbb{Z}_{8o}\frac{\partial\phi^o}{\partial\phi}u_4\mathbb{Y}_2-\mathbb{Z}_o\mathbb{Z}_3\mathbb{Y}_1\right)+[({}^v r_x^0+{}^v r_x^{01})\pi^{y^v}_{z^1}-({}^v r_y^0+{}^v r_y^{01})\pi^{z^y}_{z^1}]\mathbb{Z}_{\kappa+1}+{}^1 r_y^{1c_1}\mathbb{Z}_{\kappa+4}-{}^1 r_x^{1c_1}\mathbb{Z}_{\kappa+5},$$

$$\mathbb{Z}_{\nu+1}=\mathbb{Z}_{\mu+4}{}^1 r_z^{1c_1}-\mathbb{Z}_{\mu+7}{}^1 r_y^{1c_1}-{}^v r_z^{01}\pi^{y^v}_{x^1}+{}^v r_y^{01}\pi^{z^v}_{x^1}, \qquad \mathbb{Z}_{\nu+4}=\pi^{x^v}_{x^1}, \qquad \mathbb{Z}_{\mu+1}=\pi^{x^v}_{x^1},$$

$$\mathbb{Z}_{\nu+2}=\mathbb{Z}_{\mu+5}{}^1 r_z^{1c_1}-\mathbb{Z}_{\mu+8}{}^1 r_y^{1c_1}+{}^v r_z^{01}\pi^{x^v}_{x^1}+{}^v r_x^{01}\pi^{z^v}_{x^1}, \qquad \mathbb{Z}_{\nu+5}=\pi^{x^v}_{x^1}, \qquad \mathbb{Z}_{\mu+2}=\pi^{y^v}_{x^1},$$

$$\mathbb{Z}_{\nu+3}=\mathbb{Z}_{\mu+6}{}^1 r_z^{1c_1}-\mathbb{Z}_{\mu+9}{}^1 r_y^{1c_1}-{}^v r_y^{01}\pi^{x^v}_{x^1}+{}^v r_x^{01}\pi^{y^v}_{x^1}, \qquad \mathbb{Z}_{\nu+6}=\pi^{z^v}_{x^1}, \qquad \mathbb{Z}_{\mu+3}=\pi^{z^v}_{x^1},$$

$$\mathbb{Z}_{\nu+7}=\mathbb{Z}_{\mu+7}{}^1 r_x^{1c_1}-\mathbb{Z}_{\mu+1}{}^1 r_z^{1c_1}-{}^v r_z^{01}\pi^{y^v}_{y^1}+{}^v r_y^{01}\pi^{z^v}_{y^1}, \qquad \mathbb{Z}_{\nu+10}=\pi^{x^v}_{y^1}, \qquad \mathbb{Z}_{\mu+4}=\pi^{x^v}_{y^1},$$

$$\mathbb{Z}_{\nu+8}=\mathbb{Z}_{\mu+8}{}^1 r_x^{1c_1}-\mathbb{Z}_{\mu+2}{}^1 r_z^{1c_1}+{}^v r_z^{01}\pi^{x^v}_{y^1}+{}^v r_x^{01}\pi^{z^v}_{y^1}, \qquad \mathbb{Z}_{\nu+11}=\pi^{y^v}_{y^1}, \qquad \mathbb{Z}_{\mu+5}=\pi^{y^v}_{y^1},$$

$$\mathbb{Z}_{\nu+9}=\mathbb{Z}_{\mu+9}{}^1 r_x^{1c_1}-\mathbb{Z}_{\mu+3}{}^1 r_z^{1c_1}-{}^v r_y^{01}\pi^{x^v}_{y^1}+{}^v r_x^{01}\pi^{z^v}_{y^1}, \qquad \mathbb{Z}_{\nu+12}=\pi^{z^v}_{y^1}, \qquad \mathbb{Z}_{\mu+6}=\pi^{z^v}_{y^1},$$

$$\mathbb{Z}_{\nu+13}=\mathbb{Z}_{\mu+1}{}^1 r_y^{1c_1}-\mathbb{Z}_{\mu+4}{}^1 r_x^{1c_1}-{}^v r_z^{01}\pi^{y^v}_{z^1}+{}^v r_y^{01}\pi^{z^v}_{z^1}, \qquad \mathbb{Z}_{\nu+16}=\pi^{x^v}_{z^1}, \qquad \mathbb{Z}_{\mu+7}=\pi^{x^v}_{z^1},$$

$$\mathbb{Z}_{\nu+14}=\mathbb{Z}_{\mu+2}{}^1 r_y^{1c_1}-\mathbb{Z}_{\mu+5}{}^1 r_x^{1c_1}+{}^v r_z^{01}\pi^{x^v}_{z^1}+{}^v r_x^{01}\pi^{z^v}_{z^1}, \qquad \mathbb{Z}_{\nu+17}=\pi^{y^v}_{z^1}, \qquad \mathbb{Z}_{\mu+8}=\pi^{y^v}_{z^1},$$

$$\mathbb{Z}_{\nu+15}=\mathbb{Z}_{\mu+3}{}^1 r_y^{1c_1}-\mathbb{Z}_{\mu+6}{}^1 r_x^{1c_1}-{}^v r_y^{01}\pi^{x^v}_{z^1}+{}^v r_x^{01}\pi^{y^v}_{z^1}, \qquad \mathbb{Z}_{\nu+18}=\pi^{z^v}_{z^1}, \qquad \mathbb{Z}_{\mu+9}=\pi^{z^v}_{z^1},$$

$$\mathbf{A}_i = ({}^{i}_{i-1}\mathbf{R}) \cdot \begin{bmatrix} & 0 \\ \mathbf{A}_{i-1} & 0 \\ & 1 \end{bmatrix}, \qquad \mathbf{A}_i^+ = \begin{bmatrix} & 0 \\ \mathbf{A}_i & 0 \\ & 1 \end{bmatrix} \cdot [u_1 \;\; u_2 \;\; u_5 \;\; \cdots \;\; u_{i+4}]^T,$$

$$[\mathbb{X}]_i = \mathbf{L}_{c_{i-1}i} \cdot \begin{bmatrix} & 0 \\ \mathbf{A}_{i-1} & 0 \\ & 1 \end{bmatrix} + \mathbf{B}_{i-1}^T, \qquad \mathbf{B}_i = [{}^{i}_{i-1}\mathbf{R} \;\; \mathbf{L}_{ic_i}] \begin{bmatrix} [\mathbb{X}]_i^T & \mathbf{A}_i^T \\ 0\;0\;0 & 0\;0\;1 \end{bmatrix}^T,$$

$$\mathbf{B}_i^+ = \mathbf{B}_i [u_1 \;\; u_2 \;\; u_5 \;\; \cdots \;\; u_{i+4}]^T, \qquad \boldsymbol{\Gamma}_i = \frac{\partial ({}^{i}_{i-1}\mathbf{R})}{\partial q_i} \mathbf{A}_i^+ + ({}^{i}_{i-1}\mathbf{R}) \boldsymbol{\Gamma}_{i-1}, \quad [\mathbb{X}]_{\pi_i} = \mathbf{N}_{i-1}^{1\;T} + \mathbf{L}_{c_{i-1}i} \mathbf{M}_{i-1}$$

$$[\mathbb{Y}]_i^1 = \mathbf{L}_{c_{i-1}i} \boldsymbol{\Gamma}_{i-1} + [\mathbb{Y}]_{i-1}^2, \quad [\mathbb{Y}]_i^2 = \left\{ \frac{\partial ({}^{i}_{i-1}\mathbf{R})}{\partial q_i} [\mathbb{X}]_i \; [u_1 \;\; \ldots \;\; u_{i+3}]^T \right\} u_{i+4} + ({}^{i}_{i-1}\mathbf{R}) [\mathbb{Y}]_i^1 + \mathbf{L}_{ic_i} \boldsymbol{\Gamma}_i,$$

$$\boldsymbol{\Delta}_i = [\mathbb{Y}]_i^2 + \mathbf{A}_i^+ \times \mathbf{B}_i^+, \quad \mathbf{M}_i = ({}^{i}_{i-1}\mathbf{R}) \mathbf{M}_{i-1}, \quad \mathbf{N}_i^1 = ({}^{i}_{i-1}\mathbf{R}) [\mathbb{X}]_{\pi_i} + \mathbf{L}_{ic_i} \mathbf{M}_i, \quad \mathbf{N}_i^2 = ({}^{i}_{i-1}\mathbf{R}) \mathbf{N}_{i-1}^2,$$