Stability Properties of Interconnected Vehicles

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Abstract

The paper presents a methodology for analyzing the stability of formations of interconnected vehicles that are based on leader-follower relations. The methodology exploits input-to-state stability properties of basic leader-follower interconnections and builds on the propagation of these properties throughout the network to establish global stability bounds. This is formalized using the notion of *(formation ISS)*, a weaker form of stability than string or mesh stability, which relates leader input(s) to formation state errors. In this paper we focus on cyclic interconnections of vehicles and show how the ISS framework can be extended to include these structures.

1 Introduction

Interconnected systems have lately received considerable attention, motivated by recent advances in computation and communication, which provide the enabling technology for applications such as automated highway systems [1], cooperative robot reconnaissance [2] satellite clustering [3] and control of groups of unmanned vehicles [4, 5, 6].

Most analysis methods focus on network architecture and coordination between the systems. In the behavior based approach [2] the group behavior emerges as a combination of the behavior of each group member, selected among a set of primitive actions. Another approach focuses on maintaining a certain group configuration and forces each agent to behave as a particle in a rigid virtual structure [7]. The leader-follower approach [8, 9] distinguishes a designated leader which the other agents follow either directly or indirectly.

Another research direction aims at establishing the stability of the interconnected system. String stability [1] and mesh stability [10], the latter being the generalization of the former in more than two dimensions express the property of the interconnection to damp disturbances as they propagate through the network. This property relies on exponential stability of each individual unforced system and globally Lipschitz continuity of the right hand sides of the system dynamic equations.

Our approach addresses stability from a different perspective and imposes less stringent conditions. We focus on the way error signals propagate through the network and derive measures that characterize the network topology in terms of stability with respect to reference signals. Performance is analyzed using the notion of *formation Input-to-State-Stability (ISS)* [11, 12], based on the property that ISS is preserved in many interesting system interconnections [13, 14]. Exploiting the interconnection properties, we are able to propagate

ISS from the level of a basic interconnection to the whole network. In this paper we offer less conservative results for the cascade, parallel and multiple-leader interconnections than those of our earlier work [11, 12] and we consider for the first time the case of formation structures with cycles. The rest of the paper is organized as follows: section 2 provides the graph theoretic framework for interconnections of vehicles and introduces the notion of formation input-to-state stability. Section 3 establishes the ISS properties of some primitive formations. These primitives are used as building blocks for more complex formation structures. Propagation of formation ISS is described in Section 4 by a means of an algorithm for ISS gains computation. Section 5 summarizes the results.

2 Formation Control Graphs

Agent interaction [6, 8, 9] has been successfully represented in literature using graph theoretic notation. In this paper, the leader-follower interconnections between the agents of a formation are realized by state feedback laws and modeled by means of a directed *formation control graph*.

Definition 2.1 (Formation Control Graph). *A formation control graph $F_c = (V, E, D)$ is a directed graph that consists of:*

- *A finite set $V = \{v_1, \ldots, v_p\}$ of p vertices and a mapping $v_i \mapsto T\mathbb{R}^n \times \mathbb{R}^m$ that assigns to each vertex a control system describing the dynamics of a particular agent: $\dot{x}_i = A_i x_i + B_i u_i$, where $x_i \in \mathbb{R}^n$ is the state of the agent, $u_i \in \mathbb{R}^m$ is the agent control input and (A_i, B_i) is a controllable pair.*

- *A binary relation $E \subset V \times V$ representing a leader-follower relation between agents, implemented by a linear feedback control law $u_i(x_i, x_{j_1}, \ldots, x_{j_r})$ such that $(v_{j_k}, v_i) \in E$, with $r \leq n$ being the in-degree of vertex i.*

- *A finite set D of formation specifications indexed by the set E, $D = \{d_{ij}\}_{(v_j,v_i)\in E}$. For each edge (v_j, v_i), the vector $d_{ij} \in \mathbb{R}^n$, denotes the desired relative position between agent i and agent of vertex j.*

To every follower i such that $(v_j, v_i) \in E$ we associate an error vector that expresses the deviation from the specification prescribed for that interconnection: $\tilde{x}_i \triangleq x_j - x_i - d_{ij} \in \mathbb{R}_n$. The formation error $\tilde{x}$ is defined as the vector $\tilde{x} \triangleq \left[\tilde{x}_1^T \cdots \tilde{x}_i^T \cdot\right]^T_{v_i \in V}$. Assuming that $\{u_L\}$, $L \subset E$ are inputs to the formation leaders, we define the notion of formation input-to-state stability:

Definition 2.2 (Formation Input-to-State Stability). *A formation is called input-to-state stable if there is a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial formation error $x(0)$ and for any bounded inputs of the formation leader(s) $\{u_L(t)\}$*

the evolution of the formation error satisfies:

$$\|x(t)\| \le \beta(\|x(0)\|_2, t) + \sum_{\ell \in L} \gamma_\ell(\sup_{[0,t]} \|u_\ell\|) \tag{2.1}$$

It is known [13, 14] that certain interconnections of ISS systems preserve the ISS property. We decompose the formation graph to a small number of primitive subgraphs of depth two, which serve as building blocks: the cascade interconnection of three agents (Figure 1), the parallel interconnection of four agents (Figure 2), the multiple-leader interconnection (Figure 3) and the cyclic interconnection (Figure 4. Graphs that can be decomposed into these subgraphs are representative of a fairly broad class of formations.

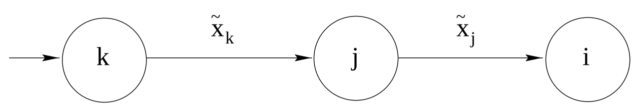

Figure 1: Cascade interconnection of agents.

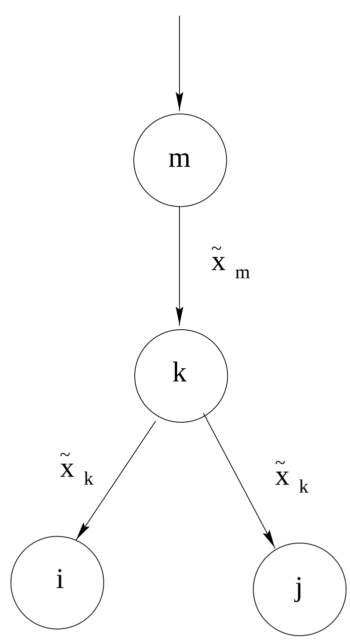

Figure 2: Parallel interconnection of agents.

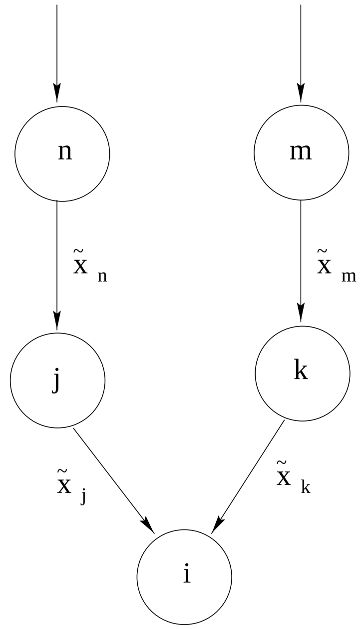

Figure 3: Multiple leader interconnection of agents.

3 ISS of Basic Interconnections

Consider two agents, i and j where i is assigned to follow j (Figure 1). Let their dynamics be expressed as:

$$\dot{x}_i = A_i x_i + B_i u_i \tag{3.2a}$$

$$\dot{x}_j = A_j x_j + B_j u_j \tag{3.2b}$$

where A_i, A_j are real $n \times n$ matrices and B_i, B_j of appropriate dimensions. Suppose that the desired position of i with respect to j is : $x_i^r = x_j - d_{ij}$. The error for agent j can be defined as: $\tilde{x}_i = x_j - d_{ij} - x_i$, where d_{ij} is a constant n-dimensional vector.

For $x_i^r = x_j - d_{ij}$ to be an equilibrium of the closed loop follower dynamics: $\dot{x}_i = A_i x_i + B_i u_i$, it should hold that $A_i x_i^r \in \mathcal{R}(B_i)$. Suppose there exists e_{ij} such that $B_i e_i = -A_i x_i^r$. Let the control law for the follower be of the form:

$$u_i = K_i(x_j - x_i - d_{ij}) + e_{ij} \tag{3.3}$$

which yields the follower dynamics:

$$\dot{x}_i = (A_i - B_i K_i)(x_i - x_j + d_{ij}).$$

The error dynamics for follower i are:

$$\dot{\tilde{x}}_i = (A_i - B_i K_i)\tilde{x}_i + \dot{x}_j \tag{3.4}$$

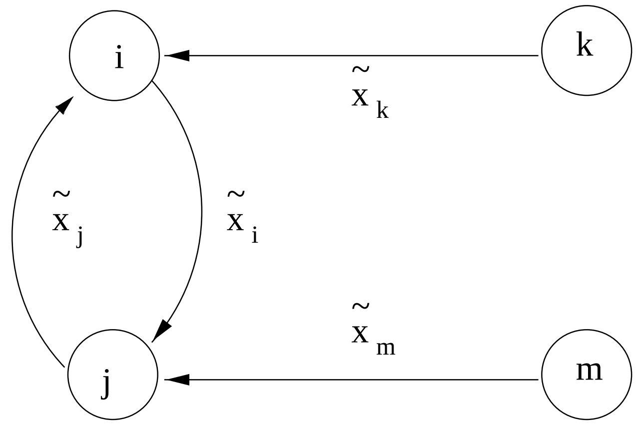

Figure 4: A cyclic interconnection

Proposition 3.1. *Let the dynamics of follower i and leader j be described by* (3.2) *and follower i is driven by a linear control law* (3.3)*. Then the closed loop system is ISS with respect to the leader's velocity and the ISS gains are given by:*

$$\hat{\beta}_i = \left(\frac{\lambda_M[P_i]}{\lambda_m[P_i]}\right)^{\frac{1}{2}} \qquad \hat{\gamma}_{i_j} = \frac{2(\lambda_M[P_i])^{\frac{3}{2}}}{(\lambda_m[P_i])^{\frac{1}{2}}\theta} \tag{3.5}$$

Proof. Given the controllability assumption, K_i can be chosen so that $A_i - B_iK_i$ is Hurwitz. Then the solution of the Lyapunov equation: $P_i(A_i - B_iK_i) + (A_i - B_iK_i)^T P_i = -I$, provides a symmetric and positive definite matrix P_i and a natural Lyapunov function candidate $V_i = \tilde{x}_i^T P_i \tilde{x}_i$ for the interconnection dynamics (3.4) that satisfies: $\lambda_m[P_i] \|\tilde{x}_i\| \leq V_i \leq \lambda_M[P_i] \|\tilde{x}_i\|$, where $\lambda_m[\cdot]$ and $\lambda_M[\cdot]$ denote the minimum and maximum eigenvalue of a given matrix, respectively. Then,

$$\dot{V}_i \leq -\|\tilde{x}_i\|^2 + 2\lambda_M(P_i)\|\tilde{x}_i\|\|\dot{x}_j\| \leq -(1-\theta)\|\tilde{x}_i\|^2 \leq 0, \forall \|\tilde{x}_i\| \geq \frac{2\lambda_M[P_i]}{\theta}\|\dot{x}_j\|$$

where $\theta \in (0,1)$. Viewing (3.4) as a perturbed system:

$$\|\tilde{x}_i(t)\| \leq \left(\frac{\lambda_M[P_i]}{\lambda_m[P_i]}\right)^{\frac{1}{2}} \|\tilde{x}_i(0)\| \, \mathrm{e}^{-\frac{1-\theta}{2\lambda_M[P_i]}t} + \frac{2(\lambda_M[P_i])^{\frac{3}{2}}}{(\lambda_m[P_i])^{\frac{1}{2}}\theta} \sup\left(\|\dot{x}_j\|\right) \tag{3.6}$$

Equation (3.6) implies that (3.4) is input-to-state stable with respect to $\dot{x}_j$ as input and

$$\beta_i(r,t) = r\hat{\beta}_i \mathrm{e}^{-\frac{1-\theta}{2\lambda_M[P_i]}t} \qquad \gamma_{i_j}(r) = \hat{\gamma}_{i_j} r$$

where

$$\hat{\beta}_i = \left(\frac{\lambda_M[P_i]}{\lambda_m[P_i]}\right)^{\frac{1}{2}} \qquad \hat{\gamma}_{i_j} = \frac{2(\lambda_M[P_i])^{\frac{3}{2}}}{(\lambda_m[P_i])^{\frac{1}{2}}\theta}$$

□

Consider now five agents i, j, k, m and n and let their dynamics be given by:

$$\dot{x}_i = A_i x_i + B_i u_i \tag{3.7a}$$
$$\dot{x}_j = A_j x_j + B_j u_j \tag{3.7b}$$
$$\dot{x}_k = A_k x_k + B_k u_k \tag{3.7c}$$
$$\dot{x}_m = A_m x_m + B_m u_m \tag{3.7d}$$
$$\dot{x}_m = A_n x_n + B_n u_n \tag{3.7e}$$

The cascade interconnection of Figure 1 can be realized by control laws of the form:

$$u_r = K_r \tilde{x}_r + e_r, \quad r \in \{i, j, k\} \tag{3.8}$$

where e_i, e_j and e_k are such that: $B_i e_i = -A_i x_i^r$, $B_j e_j = -A_j x_j^r$, and $B_k e_k = -A_k x_k^r$. Then, as expected, the interconnection will be ISS. The following proposition formalizes this fact. It provides less conservative bounds compared to [11, 12] exploiting fully the linear structure of the agent dynamics.

Proposition 3.2 (Linear Cascade Interconnection). *Consider the interconnection of Figure 1 where the dynamics of the agents are given by* (3.7)*. Then the application of control laws* (3.8) *results in a closed loop $\tilde{x}_g = (\tilde{x}_i, \tilde{x}_j)$-system which is ISS with respect to $\tilde{x}_k$:*

$$\|\tilde{x}_g\| \leq \bar{\beta}_g \|\tilde{x}_g(0)\| e^{\frac{-(1-\theta)t}{4\max\{\lambda_M[P_i], \lambda_M[P_j]\}}} + \bar{\gamma}_{g_k} \sup \|\tilde{x}_k\|, \ \textit{where:}$$

$$\bar{\beta}_g \triangleq \bar{\beta}_i^2 + (1 + \bar{\beta}_i)\bar{\beta}_j \bar{\gamma}_{i_j} + \bar{\beta}_j,$$
$$\bar{\gamma}_{g_k} \triangleq (\bar{\beta}_i + 1)\bar{\gamma}_{i_j}\bar{\gamma}_{j_k} + \bar{\gamma}_{j_k}$$

with

$$\bar{\beta}_i = \left(\frac{\lambda_M[P_i]}{\lambda_m[P_i]}\right)^{\frac{1}{2}} \qquad \bar{\gamma}_{i_j} = \frac{2(\lambda_M[P_i])^{\frac{3}{2}} \lambda_M[A_j - B_j K_j]}{(\lambda_m[P_i])^{\frac{1}{2}} \theta} \tag{3.9}$$

and P_i being the solution of the Lyapunov equation: $P_i(A_i - B_i K_i) + (A_i - B_i K_i)^T P_i = -I$

Proof. Substituting (3.8) to (3.7) results in:

$$\begin{aligned}
\dot{\tilde{x}}_i &= (A_i - B_i K_i)\tilde{x}_i - (A_j - B_j K_j)\tilde{x}_j \\
\dot{\tilde{x}}_j &= (A_j - B_j K_j)\tilde{x}_j - (A_k - B_k K_k)\tilde{x}_k \\
\dot{x}_k &= -(A_k - B_k K_k)\tilde{x}_k
\end{aligned}$$

Setting (Prop. 3.1) $\bar{\beta}_i \triangleq \left(\frac{\lambda_M[P_i]}{\lambda_m[P_i]}\right)^{\frac{1}{2}} \equiv \hat{\beta}_i$, $\bar{\gamma}_{i_j} \triangleq \frac{2(\lambda_M[P_i])^{\frac{3}{2}} \lambda_M[A_j - B_j K_j]}{(\lambda_m[P_i])^{\frac{1}{2}} \theta}$ and similarly for j,:

$$\|\tilde{x}_i\| \leq \bar{\beta}_i \|\tilde{x}_i(0)\| e^{-\frac{1-\theta}{2\lambda_M[P_i]}t} + \bar{\gamma}_{i_j} \sup \|\tilde{x}_j\| \tag{3.10}$$

$$\|\tilde{x}_j\| \leq \bar{\beta}_j \|\tilde{x}_j(0)\| e^{-\frac{1-\theta}{2\lambda_M[P_j]}t} + \bar{\gamma}_{j_k} \sup \|\tilde{x}_k\| \tag{3.11}$$

Setting the initial time to $\frac{t}{2}$ in (3.10):

$$\|\tilde{x}_i(t)\| \leq \bar{\beta}_i \|\tilde{x}_i(t/2)\| e^{\frac{-(1-\theta)t}{4\lambda_M[P_i]}} + \bar{\gamma}_{i_j} \sup_{[t/2,t]} \|\tilde{x}_j\| \tag{3.12}$$

By (3.11), the last term is bounded by: $\sup_{[t/2,t]} \|\tilde{x}_j\| \leq \bar{\beta}_j \|\tilde{x}_j(0)\| e^{-\frac{1-\theta}{4\lambda_M[P_j]}t} + \bar{\gamma}_{j_k} \sup_{[0,t]} \|\tilde{x}_k\|$. On the other hand, for the time interval $[0, t/2]$, (3.10) gives: $\|\tilde{x}_i(t/2)\| \leq \bar{\beta}_i \|\tilde{x}_i(0)\| e^{-\frac{1-\theta}{4\lambda_M[P_i]}t} + \bar{\gamma}_{i_j} \sup_{[0,t/2]} \|\tilde{x}_j\|$ and so, substituting in (3.12) and combining with (3.11):

$$\|\tilde{x}_g(t)\| \leq ((\bar{\beta}_i+1)\bar{\gamma}_{i_j}\bar{\gamma}_{j_k}+\bar{\gamma}_{j_k}) \sup \|\tilde{x}_k\| + (\bar{\beta}_i^2+(1+\bar{\beta}_i)\bar{\beta}_j\bar{\gamma}_{i_j}+\bar{\beta}_j) \|\tilde{x}_g(0)\| e^{\frac{-(1-\theta)t}{4\max\{\lambda_M[P_i],\lambda_M[P_j]\}}}$$

□

The parallel and multiple leader interconnections of Figures 2 and 3 can also be made ISS by appropriate choice of control gains. The proofs of the following propositions are in the same spirit with that of Proposition 3.2 and will be omitted for brevity.

The parallel interconnection of Figure 2 is realized by application of the control laws:

$$u_i = K_i\tilde{x}_i + e_i \qquad u_j = K_j\tilde{x}_j + e_j \tag{3.13}$$

$$u_k = K_k\tilde{x}_k + e_k \qquad u_m = K_m\tilde{x}_m + e_m \tag{3.14}$$

where $B_ie_i = -A_ix_i^r$, $B_je_j = -A_jx_j^r$, $B_ke_k = -A_kx_k^r$, and $B_me_m = -A_mx_m^r$ and can be shown to be input-to-state stable:

Proposition 3.3 (Linear Parallel Interconnection). *Consider the parallel interconnection of Figure 2 where the dynamics of the agents are given by* (3.7)*. Under* (3.14) *the closed loop system for* $\tilde{x}_g = (\tilde{x}_i, \tilde{x}_j, \tilde{x}_k)$ *is ISS with respect to* $\tilde{x}_m$*:*

$$\|\tilde{x}_g\| \leq \bar{\beta}_g \|\tilde{x}_g(0)\| e^{\frac{-(1-\theta)t}{4\max\{\lambda_M[P_i],\lambda_M[P_j],\lambda_M[P_k]\}}} + \bar{\gamma}_{g_m} \sup \|\tilde{x}_m\|, \qquad \textit{where:}$$
$$\bar{\beta}_g = \bar{\beta}_i^2 + \bar{\beta}_j^2 + ((1+\bar{\beta}_i)\bar{\gamma}_{i_k} + (1+\bar{\beta}_j)\bar{\gamma}_{j_k} + 1)\bar{\beta}_k$$
$$\bar{\gamma}_{g_k} = ((\bar{\beta}_i+1)\bar{\gamma}_{i_k} + (\bar{\beta}_j+1)\bar{\gamma}_{j_k} + 1)\bar{\gamma}_{k_m}$$

$\bar{\beta}_i$ *and* $\bar{\gamma}_{i_j}$ *are given by* (3.9)*, and* P_i *is the solution of:* $P_i(A_i - B_iK_i) + (A_i - B_iK_i)^T P_i = -I$

In a multiple leader interconnection, an agent i is assigned to follow two different leaders, say j and k. Let the specification for agent i be to follow part of the state of j and part of the state of k: $x_i^r \triangleq S_j(x_j - d_{ij}) + S_k(x_k - d_{ik})$, where d_{ij}, d_{ik} are offset vectors, and S_j, S_k are selection matrices of zeros and ones such that $\text{rank}[S_j] + \text{rank}[S_k] = \dim(x_i)$. Suppose agents j and k are required to follow $x_j^r = x_n - d_{jn}$ and $x_k^r = x_m - d_{km}$, respectively. Then the errors are defined as: $\tilde{x}_i = S_j(x_j - d_{ij}) + S_k(x_k - d_{ik}) - x_i$, $\tilde{x}_j = x_j^r - x_j$ and $\tilde{x}_k = x_k^r - x_k$. This interconnection can then be realized by the control laws:

$$\begin{aligned} u_i &= e_i + K_i\tilde{x}_i \qquad & u_j &= e_j + K_j\tilde{x}_j \\ u_k &= e_k + K_k\tilde{x}_k \qquad & u_m &= e_m + K_m\tilde{x}_m \\ u_n &= e_n + K_n\tilde{x}_n \end{aligned} \tag{3.15}$$

where e_i, e_j and e_k satisfy $B_ie_i = -A_ix_i^r$, $B_je_j = -A_jx_j^r$, $B_ke_k = -A_kx_k^r$, $B_me_m = -A_mx_m^r$ and $B_ne_n = -A_nx_n^r$.

Proposition 3.4 (Linear Multiple-Leader Interconnection). *Let the dynamics of agents i, j, k, n and m be given by* (3.7)*. Under* (3.15)*, the closed loop system for $\tilde{x}_g = (\tilde{x}_i, \tilde{x}_j, \tilde{x}_k)$ is ISS with respect to $\tilde{x}_m$ and $\tilde{x}_n$:*

$$
\begin{aligned}
&\|\tilde{x}_g\| \le \bar{\beta}_g \|\tilde{x}_g(0)\| e^{-\frac{1-\theta}{4\max\{\lambda_M[P_i],\lambda_M[P_j],\lambda_M[P_k]\}}t} + \bar{\gamma}_{g_n} \sup \|\tilde{x}_n\| + \bar{\gamma}_{g_m} \sup \|\tilde{x}_m\|, \qquad \textit{where:}\\
&\quad \bar{\beta}_g = \bar{\beta}_i^2 + (1+\bar{\beta}_i)(\bar{\beta}_j\bar{\gamma}_{i_j} + \bar{\beta}_k\bar{\gamma}_{i_k}) + \bar{\beta}_j + \bar{\beta}_k\\
&\quad \bar{\gamma}_{g_n} = \bar{\gamma}_{i_j}\bar{\gamma}_{j_n}(1+\bar{\beta}_i) + \bar{\gamma}_{j_n}\\
&\quad \bar{\gamma}_{g_m} = \bar{\gamma}_{i_k}\bar{\gamma}_{k_m}(1+\bar{\beta}_i) + \bar{\gamma}_{k_m}
\end{aligned}
$$

$\bar{\gamma}_{i_k}$ given by (3.9)*, and P_i is the solution of: $P_i(A_i - B_iK_i) + (A_i - B_iK_i)^TP_i = -I$*

Cyclic interconnections can be reduced to the one depicted in Figure 4 by combining cascades. It can be shown that under reasonable assumptions, the cyclic interconnection can be input-to-state stable. The result is based on a version of small gain theorem [14] and requires that the signal flowing through the cyclic path is attenuating.

Let the dynamics of the agents be given by (3.7), and define the control laws as follows:

$$
\begin{aligned}
u_i &= K_i\tilde{x}_i + e_i \qquad u_j = K_j\tilde{x}_j + e_j\\
u_k &= K_k\tilde{x}_k + e_k \quad u_m = K_m\tilde{x}_m + e_m
\end{aligned} \tag{3.16}
$$

Since $(A_i - B_iK_i)$ and $(A_j - B_jK_j)$ are Hurwitz,

$$
\begin{aligned}
\|\tilde{x}_i\| &\le \bar{\beta}_i \|\tilde{x}_i(0)\| e^{\frac{-(1-\theta)t}{2\lambda_M[P_i]}} + \bar{\gamma}_{i_j} \sup \|\tilde{x}_j\| + \bar{\gamma}_{i_k} \sup \|\tilde{x}_k\|\\
\|\tilde{x}_j\| &\le \bar{\beta}_j \|\tilde{x}_j(0)\| e^{\frac{-(1-\theta)t}{2\lambda_M[P_j]}} + \bar{\gamma}_{j_m} \sup \|\tilde{x}_m\| + \bar{\gamma}_{j_i} \sup \|\tilde{x}_i\|
\end{aligned}
$$

with $\bar{\beta}_i$, $\bar{\beta}_j$, $\bar{\gamma}_{i_j}$, $\bar{\gamma}_{j_m}$, $\bar{\gamma}_{i_j}$ and $\bar{\gamma}_{j_i}$ given by (3.9) and P_i, P_j being the solutions of $P_i(A_i - B_iK_i) + (A_i - B_iK_i)^TP_i = -I$ and $P_j(A_j - B_jK_j) + (A_j - B_jK_j)^TP_j = -I$, respectively. For notational brevity we will denote the supremum of a norm of a signal by its $\mathcal{L}_\infty$ norm: $\sup_{t\ge0}\|z(t)\| \equiv \|z(t)\|_\infty$. Using an alternative characterization of ISS [14]:

$$\|\tilde{x}_i\| \le \max\{\bar{\beta}_i \|\tilde{x}_i(0)\|, \bar{\gamma}_{i_j} \|\tilde{x}_j\|_\infty, \bar{\gamma}_{i_k} \|\tilde{x}_k\|_\infty\} \tag{3.17}$$

$$\lim_{t\uparrow\infty} \|\tilde{x}_i\|_\infty \le \max\{\bar{\gamma}_{i_j} \lim_{t\uparrow\infty} \|\tilde{x}_j\|_\infty, \bar{\gamma}_{i_k} \lim_{t\uparrow\infty} \|\tilde{x}_k\|_\infty\} \tag{3.18}$$

$$\|\tilde{x}_j\| \le \max\{\bar{\beta}_j \|\tilde{x}_j(0)\|, \bar{\gamma}_{j_i} \|\tilde{x}_i\|_\infty, \bar{\gamma}_{j_m} \|\tilde{x}_m\|_\infty\} \tag{3.19}$$

$$\lim_{t\uparrow\infty} \|\tilde{x}_j\|_\infty \le \max\{\bar{\gamma}_{j_i} \lim_{t\uparrow\infty} \|\tilde{x}_i\|_\infty, \bar{\gamma}_{j_m} \lim_{t\uparrow\infty} \|\tilde{x}_m\|_\infty\} \tag{3.20}$$

Proposition 3.5 (Linear Cyclic Interconnection). *Consider the cyclic interconnection of Figure 4 where the dynamics of the agents are given by* (3.7)*. Under* (3.16) *and if $\bar{\gamma}_{i_j}\bar{\gamma}_{j_i} < 1$*

where $\bar{\gamma}_{i_j}$ *and* $\bar{\gamma}_{j_i}$ *are given by* (3.9) *then the closed loop system for* $\tilde{x}_g = (\tilde{x}_i, \tilde{x}_j)$ *is ISS with respect to* $\tilde{x}_m$ *and* $\tilde{x}_k$*:*

$$\|\tilde{x}_g\|_\infty \le \max\{\bar{\beta}_g \|\tilde{x}_g(0)\|, \bar{\gamma}_{g_k} \|\tilde{x}_k\|_\infty, \bar{\gamma}_{g_m} \|\tilde{x}_m\|_\infty\}$$
$$\lim_{t\uparrow\infty} \|\tilde{x}_g\|_\infty \le \max\{\bar{\gamma}_{g_k} \lim_{t\uparrow\infty} \|\tilde{x}_k\|_\infty, \bar{\gamma}_{g_m} \lim_{t\uparrow\infty} \|\tilde{x}_m\|_\infty\}$$

The proposition practically asserts that if the input signal that flows through the cycle attenuates, then the cyclic interconnection is ISS. This can be thought of as a special case of the small gain theorem, tailored for feedback interconnections of linear systems in which state errors of one system are inputs to another.

Proof. Equations (3.17) and (3.19) imply:

$$\|\tilde{x}_i\|_\infty \le \max\{\bar{\beta}_i \|\tilde{x}_i(0)\|, \bar{\gamma}_{i_j} \|\tilde{x}_j\|_\infty, \bar{\gamma}_{i_k} \|\tilde{x}_k\|_\infty\} \tag{3.21a}$$
$$\|\tilde{x}_j\|_\infty \le \max\{\bar{\beta}_j \|\tilde{x}_j(0)\|, \bar{\gamma}_{j_i} \|\tilde{x}_i\|_\infty, \bar{\gamma}_{j_m} \|\tilde{x}_m\|_\infty\} \tag{3.21b}$$

Since $\bar{\gamma}_{i_j}\bar{\gamma}_{j_i} < 1$, (3.21b) and (3.21a) yield:

$$\|\tilde{x}_i\|_\infty \le \max\{\bar{\beta}_i \|\tilde{x}_i(0)\|, \bar{\gamma}_{i_j}\bar{\beta}_j \|\tilde{x}_j(0)\|, \bar{\gamma}_{i_j}\bar{\gamma}_{j_m} \|\tilde{x}_m\|_\infty, \bar{\gamma}_{i_k} \|\tilde{x}_k\|_\infty\}$$
$$\|\tilde{x}_j\|_\infty \le \max\{\bar{\beta}_j \|\tilde{x}_j(0)\|, \bar{\gamma}_{j_i}\bar{\beta}_i \|\tilde{x}_i(0)\|, \bar{\gamma}_{j_i}\bar{\gamma}_{i_k} \|\tilde{x}_k\|_\infty, \bar{\gamma}_{j_m} \|\tilde{x}_m\|_\infty\}$$

Observing that $\|\tilde{x}_g\|_\infty \le \max\{2\|\tilde{x}_i\|_\infty, 2\|\tilde{x}_j\|_\infty\}$,

$$\|\tilde{x}_g\|_\infty \le \max\{4\max\{\bar{\beta}_i, \bar{\gamma}_{j_i}\bar{\beta}_i, \bar{\gamma}_{i_j}\bar{\beta}_j, \bar{\beta}_j\} \|\tilde{x}_g(0)\|, 2\max\{\bar{\gamma}_{i_k}, \bar{\gamma}_{j_i}\bar{\gamma}_{i_k}\} \|\tilde{x}_k\|_\infty, 2\max\{\bar{\gamma}_{i_j}\bar{\gamma}_{j_m}, \bar{\gamma}_{j_m}\} \|\tilde{x}_m\|_\infty\} \tag{3.22}$$

Substituting (3.20) into (3.18):

$$\limsup_{t\uparrow\infty} \|\tilde{x}_i\| \le \max\{\bar{\gamma}_{i_j}\bar{\gamma}_{j_i} \lim_{t\uparrow\infty} \|\tilde{x}_i\|_\infty, \bar{\gamma}_{i_j}\bar{\gamma}_{j_m} \lim_{t\uparrow\infty} \|\tilde{x}_m\|_\infty, \bar{\gamma}_{i_k} \lim_{t\uparrow\infty} \|\tilde{x}_k\|_\infty\}$$

which, since $\bar{\gamma}_{i_j}\bar{\gamma}_{j_i} < 1$ becomes

$$\lim_{t\uparrow\infty} \|\tilde{x}_i\|_\infty \le \max\{\bar{\gamma}_{i_j}\bar{\gamma}_{j_m} \lim_{t\uparrow\infty} \|\tilde{x}_m\|_\infty, \bar{\gamma}_{i_k} \lim_{t\uparrow\infty} \|\tilde{x}_k\|_\infty\} \tag{3.23}$$

Similarly, replacing the bound in (3.18) into (3.20):

$$\lim_{t\uparrow\infty} \|\tilde{x}_j\|_\infty \le \max\{\bar{\gamma}_{j_i}\bar{\gamma}_{i_k} \lim_{t\uparrow\infty} \|\tilde{x}_k\|_\infty, \bar{\gamma}_{j_m} \lim_{t\uparrow\infty} \|\tilde{x}_m\|_\infty\} \tag{3.24}$$

Combining (3.23) and (3.24):

$$\lim_{t\uparrow\infty} \|\tilde{x}_g\|_\infty \le \max\{2\max\{\bar{\gamma}_{j_i}\bar{\gamma}_{i_k}, \bar{\gamma}_{i_k}\} \lim_{t\uparrow\infty} \|\tilde{x}_k\|_\infty, 2\max\{\bar{\gamma}_{i_j}\bar{\gamma}_{j_m}, \bar{\gamma}_{j_m}\} \lim_{t\uparrow\infty} \|\tilde{x}_m\|_\infty\} \tag{3.25}$$

Inequalities (3.25) and (3.22) are necessary and sufficient for the ISS of the feedback interconnection (3.7) under (3.16). The linear ISS gains would be:

$$\bar{\beta}_g \triangleq 4\max\{\bar{\beta}_i, \bar{\gamma}_{j_i}\bar{\beta}_i, \bar{\gamma}_{i_j}\bar{\beta}_j, \bar{\beta}_j\}$$
$$\bar{\gamma}_{g_k} \triangleq 2\max\{\bar{\gamma}_{j_i}\bar{\gamma}_{i_k}, \bar{\gamma}_{i_k}\}$$
$$\bar{\gamma}_{g_m} \triangleq 2\max\{\bar{\gamma}_{i_j}\bar{\gamma}_{j_m}, \bar{\gamma}_{j_m}\}$$

□

4 ISS Propagation

The propositions of the previous section indicate how basic interconnection structures which can be represented by directed graphs of depth two can be equivalently reduced to simple leader-follower interconnections of depth one. Although we do not have a formalism for decomposing arbitrary acyclic directed graphs into these primitive interconnections, we feel that the class of formation structures that can be represented as combinations of these basic interconnections is rich.

The procedure for obtaining the total ISS gains for the whole formation is recursive. Starting from the vertices of outdegree zero (terminal nodes) we continuously apply propositions 3.2 - 3.5 to obtain a sequence of graph abstractions that can reduce the whole graph into a graph of depth one, representing the leader-follower relations between the formation leaders and the whole follower group. Proposition 3.5 specifically transforms a graph with an undirected edge into an equivalent with only directed edges.

In previous work [11] we have provided an algorithm to compute the ISS gains of the formation, given the ISS gains of individual interconnections and the elements of the formation graph adjacency matrix. Before the algorithm can be applied, proposition 3.5 should be used to mode out the cycles in the graph. The algorithm that is described below is in exactly the same spirit as that in [11], only that the bounds are relaxed, reflecting the less conservative results of section 3 which exploit the linear structure:

Let a_i be the i row of the $n \times n$ adjacency matrix A of the formation graph, $F = (V, E, D)$. Define:

$$\bar{\beta}^0 \triangleq \begin{bmatrix} \bar{\beta}_1 & \cdots & \bar{\beta}_n \end{bmatrix}^T, \\ \bar{\gamma}^0 \triangleq \begin{bmatrix} \bar{\gamma}_1 & \cdots & \bar{\gamma}_n \end{bmatrix}^T$$

and for the $k+1$ iteration let

$$\bar{\beta}^{k+1} \triangleq \begin{bmatrix} \bar{\beta}_1^{k+1} & \cdots & \bar{\beta}_n^{k+1} \end{bmatrix}^T, \\ \bar{\gamma}^{k+1} \triangleq \begin{bmatrix} \bar{\gamma}_1^{k+1} & \cdots & \bar{\gamma}_n^{k+1} \end{bmatrix}^T$$

be given recursively as:

$$\bar{\gamma}^{k+1} = \bar{\gamma}^k + n_{n-k} c_\gamma^k, \\ \bar{\beta}^{k+1} = \bar{\beta}^k + n_{n-k} c_\beta^k$$

where

$$n_{n-k} = [\underbrace{0 \ \cdots \ 0}_{n-k-1} \ 1 \ \cdots \ 0]^T, \\ c_\gamma^k = a_{n-k}\bar{\beta}^k a_{n-k}\bar{\gamma}^k \bar{\gamma}_{n-k}^k + a_{n-k}\bar{\gamma}^k \bar{\gamma}_{n-k}^k, \\ c_\beta^k = a_{n-k}\bar{\beta}^k a_{n-k}\bar{\gamma}^k \bar{\beta}_{n-k}^k + a_{n-k}\bar{\gamma}^k \bar{\beta}_{n-k}^k + (a_{n-k}\bar{\beta}^k)^2$$

The algorithm terminates after at most p steps where $p = |V|$. This is due to the nilpotency of the adjacency matrix in directed acyclic graphs. It is easy to see that the depth of the graph has an adverse effect on stability: the higher the depth, the larger the formation ISS gains, a conclusion that agrees with physical intuition and experimental results on error propagation [4].

5 Conclusion

The paper presents a methodology for analyzing the stability properties of formations of vehicles which is based on the notion of input-to-state stability. The approach exploits the property of ISS to be preserved in certain types of cascade and feedback interconnections in order to propagate stability bounds from a leader-follower pair to the whole group. In this paper we specifically examine the case of cyclic interconnections and show that under some reasonable "small gain" type conditions, such interconnections can be ISS and therefore treated in the general framework of formation ISS. Moreover, we take full advantage of the linearity of the assumed vehicle dynamics to relax the stability bounds.

The methodology allows the characterization of different formation structures in terms of stability and provides formal justification for experimental data [4] concerning the effect of network topology on stability. Current research is directed towards expressing ISS properties in an algebraic graph theoretic framework that would allow further insight on how network topology is linked to stability, investigating how communication can improve stability and effectively alter network topology and how the latter can affect both stability and communication properties of the group.

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