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Analysis of Deformable Object Handling

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Abstract

A manipulated deformable object is viewed as an underactuated mechanical system. In this context controllability issues are discussed and results on the nature of the constraints and the controllability properties of an important class of deformable objects being modeled with finite elements are stated. For this class of deformable objects the results permit to circumvent the usual procedure of calculating Lie brackets to establish a base for the associated Lie algebra, and answers the question of determining the kind of constraints imposed on the system in a straightforward algebraic way. Inequality constraints associated to material strength limitations are also included.

1 Introduction

Among all kinds of material that a robot is called to manipulate, very few are actually rigid. Our world is formed mainly of deformable materials, the flexibility of which varies significantly. It becomes clear that different kinds of material can not allow a uniform treatment. Therefore, there should be a way to distinguish between manipulated objects that have different properties and follow different handling strategies, bearing in mind the individual characteristics.

Previous approaches to deformable object handling focus mainly on formulating general continuous dynamic equations for the object in a way to enhance computation or allow for certain control strategies. Sun et al. [9] followed Terzopoulos' [11] hybrid approach to deformable objects. Kosuge et al. [2] used finite elements but ignored the dynamics of the object and consentrated only on the static conditions. Wu et al. [12] approximated the distributed parameter system with a lumped parameter model. Yukawa et al. [13] investigated a vibrating flexible object and modeled it using model reduction theory.

The authors have previously modeled a deformable object being manipulated by multiple mobile manipulators [10]. They used elastodynamic equations to model the object and indicated the simplest finite element grid structure able to describe the object being handled by multiple manipulators. In this paper we generalize this approach, by introducing a framework that both includes a rich variety of mechanical systems and allows more detailed description. We consider the deformable object as an underactuated mechanical system [8] and 'discretize' the distributed parameter system using finite elements [6]. As underactuated systems, a great variety of mechanical systems can be included in this framework e.g. chains, structures with passive joints, systems with rolling contact, etc. This broadens considerably the prespective of the approach to object handling.

The rest of the paper is organized as follows: In section 2 a deformable object is described as an underactuated system and the state equations are derived. In section 3 the dynamic constraints imposed on the system are classified. The controllability properties are investigated in section 4. Constraints related to material strength limitations are included in section 5. In section 6, some examples are presented which verify the theoretical results. Finally, in section 7 the results of the paper are summarized.

2 The Underactuated System

Strictly speaking, a deformable object has infinite degrees of freedom. An attempt to simplify the problem is to 'discretize' the structure, reducing the number of its degrees of freedom to a finite countable set. A popular way is finite elements.

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The extend of discretization can depend on the individual characteristics of the material. Almost rigid materials do not require dense discretization grids; flexible ones do. Adjusting the grid, completely rigid to flexible materials can be described.

We consider the discetized deformable object as an underactuated mechanical system. Underactuated systems have less inputs than degrees of freedom [8]. The system at hand is underactuated since only a few degrees of freedom are directy controlled, namely the ones coinciding with the grasp points. The rest are staticaly and dynamically coupled to the actuated and can be regarded as passive.

The class of underactuated mechanical systems is very broad and includes systems with passive joints, flexible link robots, mobile robots, flexible link robots, space robots and a variety of other systems out of many robotics fields. This broad class motivates the investigation of new manipulation tasks, including systems of rigid bodies or combinations of rigid and deformable objects, modeled as underactuated systems.

Following the Lagrangian formulation of the dynamics of mechanical systems with n generalized coordinates $\mathbf{q} = (q_1, \dots, q_n)^T \in \mathbf{Q}$, the equations of motion can be derived

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \mathbf{F}_i \qquad \mathrm{i} = 1, \dots, \mathrm{n}$$

It is well known that the above can take the form:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{K}(\mathbf{q}) = \mathbf{B}(\mathbf{q})\mathbf{u} \qquad (1)$$

where $\mathbf{M}$ is the symmetric and positive definite inertia matrix, $\mathbf{C}$ contains the Coriolis and centrifugal terms, $\mathbf{K}$ is formed by the terms associated with gravity and elastic forces, and u is input. In the case of underactuated systems the space of generalized coordinates can be partisioned into an actuated and passive part: $\mathbf{q}^T = (\mathbf{q}_1^T, \mathbf{q}_2^T)$, and (1) can be written [8]

$$\mathbf{m}_{11}\, \ddot{\mathbf{q}}_1 + \mathbf{m}_{12}\, \ddot{\mathbf{q}}_2 + \mathbf{c}_1(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{k}_1(\mathbf{q}) = 0 \qquad (2a)$$

$$\mathbf{m}_{21}\, \ddot{\mathbf{q}}_1 + \mathbf{m}_{22}\, \ddot{\mathbf{q}}_2 + \mathbf{c}_2(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{k}_2(\mathbf{q}) = \mathbf{b}(\mathbf{q})\mathbf{u} \qquad (2b)$$

where $\mathbf{b}(\mathbf{q}) \in \Re^{m \times m}$ is assumed nonsingular.

Viewing (2a) as a set of $n - m$ dynamic constraints, a natural question to ask what kind are they. We will name them *intrinsic constraints* to distinguish them from any external imposed constraints related to object material strength or obstacle avoidance. Intrinsic constraints can either be holonomic, first order nonholonomic or second order nonholonomic. The kind of constraints imposed determines the controllability properties of the system. Therefore, it is quite important to classify these constraints before proceeding to investigating controllability properties.

2.1 Collocated Linearization and State Space Description

Certain forms of system description enhance analysis. A valuable tool for analysis is feedback linearization, which unveils the system structure. An important property of system (2a) - (2b) is that it can be partially feedback linearized with respect to the actuated degrees of freedom [8]. Indeed, by examination of (2a) it can be seen that $\mathbf{m}_{11}$ is square and nonsingular, since the original inertia matrix is positive definite. Therefore (2a) can be solved for $\ddot{\mathbf{q}}_1$

$$\ddot{\mathbf{q}}_1 = -\mathbf{m}_{11}^{-1}[\mathbf{m}_{12}\, \ddot{\mathbf{q}}_2 + \mathbf{c}_1(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{k}_1(\mathbf{q})]$$

and then substitute in (2b)

$$\bar{\mathbf{m}}(\mathbf{q})\, \ddot{\mathbf{q}}_2 + \bar{\mathbf{c}}(\mathbf{q}, \dot{\mathbf{q}}) + \bar{\mathbf{k}}(\mathbf{q}) = \mathbf{bu}$$

where

$$\bar{\mathbf{m}}(\mathbf{q}) = \mathbf{m}_{22}(\mathbf{q}) - \mathbf{m}_{21}(\mathbf{q})\, \mathbf{m}_{11}^{-1}(\mathbf{q})\, \mathbf{m}_{12}(\mathbf{q})$$
$$\bar{\mathbf{c}}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{c}_2(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{m}_{21}(\mathbf{q})\, \mathbf{m}_{11}^{-1}(\mathbf{q})\, \mathbf{c}_1(\mathbf{q}, \dot{\mathbf{q}})$$
$$\bar{\mathbf{k}}(\mathbf{q}) = \mathbf{k}_2(\mathbf{q}) - \mathbf{m}_{21}(\mathbf{q})\, \mathbf{m}_{11}^{-1}(\mathbf{q})\, \mathbf{k}_1(\mathbf{q})$$

Now, using the linearizing feedback

$$\mathbf{u} = \mathbf{b}^{-1}[\bar{\mathbf{m}}(\mathbf{q})\, \mathbf{v} + \bar{\mathbf{c}}(\mathbf{q}, \dot{\mathbf{q}}) + \bar{\mathbf{k}}(\mathbf{q})] \qquad (3)$$

the system (2a) - (2b) takes the form

$$\ddot{\mathbf{q}}_2 = \mathbf{v} \qquad (4a)$$
$$\ddot{\mathbf{q}}_1 = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}}_2 + \mathbf{R}(\mathbf{q}, \dot{\mathbf{q}}) \qquad (4b)$$

where

$$\mathbf{J}(\mathbf{q}) = -\mathbf{m}_{11}^{-1}(\mathbf{q})\mathbf{m}_{12}(\mathbf{q})$$
$$\mathbf{R}(\mathbf{q}, \dot{\mathbf{q}}) = -\mathbf{m}_{11}^{-1}(\mathbf{q})\mathbf{c}_1(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{m}_{11}^{-1}(\mathbf{q})\mathbf{k}_1(\mathbf{q})$$

Then setting

$$\mathbf{x}_1 = \mathbf{q}_2 \in \Re^m, \quad \mathbf{x}_2 = \mathbf{q}_1 \in \Re^{n-m}, \quad \mathbf{x}_3 = \dot{\mathbf{q}}_2, \quad \mathbf{x}_4 = \dot{\mathbf{q}}_1,$$

equations (4a)-(4b) come into state space form [7]

$$\dot{\mathbf{x}}_1 = \mathbf{x}_3 \qquad (5a)$$
$$\dot{\mathbf{x}}_2 = \mathbf{x}_4 \qquad (5b)$$
$$\dot{\mathbf{x}}_3 = \mathbf{v} \qquad (5c)$$
$$\dot{\mathbf{x}}_4 = \mathbf{J}(\mathbf{x}_1, \mathbf{x}_2)\mathbf{v} + \mathbf{R}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \qquad (5d)$$

3 Constraint Classification

When the manipulated object is considered completely rigid, intrinsic constraints permit integration yielding algebraic expressions of the form

$$\mathbf{q}_1 = g(\mathbf{q}_2)$$

In this case they are **holonomic** and they can be used to eliminate a number of generalized coordinates and reduce the dimension of the state space by $2(n-m)$:

$$\dot{\mathbf{x}}_1 = \mathbf{x}_3 \qquad \dot{\mathbf{x}}_3 = \mathbf{v}$$

The second case is when the constraints are **first-order nonholonomic**. The dimension of the configuration space remains the same, but the dimension of its tangent bundle is reduced by $n-m$. The constraints can be expressed in the form

$$\mathbf{A}(\mathbf{q}) \begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} = 0 \quad \mathbf{A} \in \Re^{(n-m)\times 2n}. \tag{6}$$

A base $\mathbf{S}(\mathbf{q}) \in \Re^{(n+m)\times 2n}$ for the annihilator of $\mathbf{A}$ is formed and (1) is 'projected' into the space generated by the columns of $\mathbf{S}^T$. This results in $n+m$ state equations. The existence of $\mathbf{S}$ implies a relation [1]:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{bmatrix} = \mathbf{S}^T \boldsymbol{\eta} \qquad \boldsymbol{\eta} \in \Re^{m+n} \tag{7}$$

which can be used to obtain

$$\dot{\boldsymbol{\eta}} = \mathbf{D}(\mathbf{q}) + \mathbf{G}(\mathbf{q}, \boldsymbol{\eta})\mathbf{u} \tag{8}$$

where $\mathbf{D}$ and $\mathbf{G}$ are derived by differentiation and substitution of (7) into (1) and 'projection' of the resulting equations onto the distribution spanned by $\mathbf{S}^T$.

The last case is when these constraints are completely nonintegrable, forming a set of $n-m$ **second-order nonholonomic constraints**. A formal definition of second order nonholonomic constraints is given in [7] by defining the distribution $\mathbf{\Delta} = span\{\tau_0, \tau_j\}$ where

$$\tau_0 = \sum_{j=1}^{m} \dot{q}_{2,j} \frac{\partial}{\partial q_{2,j}} + \sum_{k=1}^{n-m} (\dot{q}_{1,k} \frac{\partial}{\partial q_{1,k}} + R_k \frac{\partial}{\partial \dot{q}_{1,k}}) + \frac{\partial}{\partial t}$$

$$\tau_j = \frac{\partial}{\partial \dot{q}_{2,j}} + \sum_{i=1}^{n-m} J_{ij} \frac{\partial}{\partial \dot{q}_{1,i}}, \quad j = 1, \dots, m.$$

Definition. *(from [7]): Consider the distribution* $\mathbf{\Delta}$ *and* $\tilde{\mathcal{C}}$ *its accessibility algebra. Let* $\tilde{\mathbf{C}}$ *the accessibility distribution generated by* $\tilde{\mathcal{C}}$*. The system* (4a)-(4b) *is said to be completely second-order nonholonomic if* $dim\tilde{\mathbf{C}}(\mathbf{x}, t) = 2n+1, \forall (\mathbf{x}, t) \in \mathbf{M} \times \Re$.

Here, the dimension of the state space is not reduced. We show that finite element models of deformable objects exhibit second-order nonholonomic constraints. The presence of second-order nonholonomic constraints is typical in the dynamic description of underactuated robots [5].

4 Controllability Issues

When k intrincic constraints are holonomic, then the system motion is restricted to an $n-k$ dimensional manifold. The system has $n-k$ degrees of freedom; the remaining k can be eliminated. Control must be restricted to this manifold of admissible motions.

If the constraints turn out to be completely first order nonholonomic then the system configuration space is not confined. Proving that these constraints are completely nonholonomic is equivalent to evaluating the dimension of the accessibility distribution of the system, $\mathbf{C}$, which should be $2n$. Then the system is accessible. Due to the presence of a drift term $\mathbf{D}$ in (8), accessibility do not imply controllability. For nonholonomic systems with drift there is no available general necessary and sufficient result for establishing complete controllability [4]. One has to resort to other forms of controllability, such as strong accessibility and STLC (small-time local controllability). For the latter there exist only sufficient conditions, but once it has been established one can utilize the manifold of equilibrium points of the drift vector to reach an arbitrary small neighborhood of the desired configuration. Systems which are STLC are not asymptotically stabilizable via time-invariant feedback; piecewise analytic, however, may be used.

If the system (4a)-(4b) is proved to be second-order nonholonomic, then it is has been proved that it is strongly accessible [7]. Moreover, there is a chance for smooth stabilization, provided that a sufficient condition for non-existence of a smooth stabilizing control law is not satisfied.

Theorem. *([7]): Assume that* $R_i(\mathbf{q}, 0) = 0, \forall \mathbf{q} \in \mathbf{Q}$*, for some* $i \in I_{n-m} = \{1, \dots, n-m\}$*. Let* $n - m \geq 1$ *and let* $(\mathbf{q}^e, 0)$ *denote an equilibrium solution. Then the second-order nonholonomic system, defined by* (4a)-(4b)*, is not asymptotically stabilizable to* $(\mathbf{q}^e, 0)$ *using time-invariant continuous (static or dynamic) state feedback law.*

If $\forall i, \exists \mathbf{q} \in \mathbf{Q} | R_i(\mathbf{q}, 0) \neq 0$, then the system could perhaps be stabilizable by continuous control law.

In the remaining of this section we will show that a class of deformable objects being modeled with finite elements share some interesting properties if they satisfy a certain condition. The discussion concentrates on finite element models for which (1) takes the form:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{B}\mathbf{u} \tag{9}$$

i.e. the characteristic matrices are independent of the generalized coordinates and speeds. The above model is standard and can be found in many finite elements textbooks [6]. The state equations derived from the above model have the form

$$\ddot{\mathbf{q}}_2 = \mathbf{v} \tag{10a}$$

$$\ddot{\mathbf{q}}_1 = \mathbf{J}\ddot{\mathbf{q}}_2 + \mathbf{R}(\mathbf{q}, \dot{\mathbf{q}}) \tag{10b}$$

We proceed with the following Lemma:

Lemma 1. *For equations (10a - 10b) it holds:*

1. $[\tau_k, ad^r_{\tau_0}\tau_j] = 0 \quad \forall r \geq 0, \quad k, j \in \{1, \ldots, m\}$
2. $ad^{1+r}_{\tau_0}\tau_j = \begin{bmatrix} A_r \begin{bmatrix} \mathbf{J}_j & e_j \end{bmatrix}^T \\ 0_{m\times 1} \\ B_r \begin{bmatrix} \mathbf{J}_j & e_j \end{bmatrix}^T \\ 0_{(m+1)\times 1} \end{bmatrix}$ *for* $r > 0$ *and* $A_r, B_r \in \Re^{(n-m)\times 1}$, *where* $\mathbf{J}_j$ *is the* j^{th} *column of* $\mathbf{J}$ *and* e_j *is the* j^{th} *base vector of* $\Re^m$.
3. *The vectors that form the vector field* $ad^{1+i}_{\tau_0}\tau_j$ *can be expressed as*

$$B_r = (-1)^{r-1}\left[\frac{\partial \mathbf{R}}{\partial \mathbf{q}_1}A_{r-1} + \frac{\partial \mathbf{R}}{\partial \dot{\mathbf{q}}_1}B_{r-1}\right]$$

$$A_r = (-1)^{r-1}B_{r-1} \qquad \textit{with}$$

$$B_1 = \frac{\partial \mathbf{R}}{\partial \mathbf{q}} + \frac{\partial \mathbf{R}}{\partial \dot{\mathbf{q}}_1}\cdot\frac{\partial \mathbf{R}}{\partial \dot{\mathbf{q}}} \qquad A_1 = \frac{\partial \mathbf{R}}{\partial \dot{\mathbf{q}}}$$

Proof. (1) can be proved by noticing that if we set $\bar{q} = \begin{bmatrix}\mathbf{q}^T & \dot{\mathbf{q}}^T & t\end{bmatrix}^T$ then $\frac{\partial \tau_j}{\partial \bar{q}} = 0$. (2) can be proved by straightforward calculation, taking into account that $\mathbf{J}$ is independent of $\bar{q}$ and $\mathbf{R}$ does not contain quadratic terms in neither $\mathbf{q}$ or $\dot{\mathbf{q}}$. For (3), note the special structure of $\frac{\partial \tau_0}{\partial \bar{q}}$ and that $\frac{\partial ad^i_{\tau_0}\tau_j}{\partial \bar{q}} = 0$. □

Our main result follows:

Proposition 1. *If for system (9) all matrices in the sequence*

$$\begin{bmatrix} A_1 \\ 0 \\ B_1 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{I}_{m\times m} \end{bmatrix} \cdots \begin{bmatrix} A_{\frac{2(n-m)}{m}} \\ 0 \\ B_{\frac{2(n-m)}{m}} \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{I}_{m\times m} \end{bmatrix}$$

have full rank, (9) is second-order nonholonomic.

Proof. The proof follows immediately from the above Lemma and from Definition, by noting that this sequence is directly accossiated with the filtration of the accessibility distribution $\bar{\mathbf{C}}$. The filtration is regular with relative growth vector $s = (m+1, m, \underbrace{m, \ldots, m}_{r})$. Therefore the dimension of the distribution is finally $2m + 1 + \frac{2(n-m)}{m}m = 2n$ □

Being second-order nonholonomic, the finite element model (9) is also **strongly accessible** [7]. Moreover, a careful investigation shows that it **does not satisfy the sufficient condition for STLC** presented in [3], since $\mathbf{J}$ is constant and $\mathbf{R}$ does not have a term which is quadratic in velocities. This of course does not mean that the system is not STLC. Finally, at the equilibrium the system does not satisfy the condition $R(\mathbf{q}^e, 0) = 0 \quad \forall \mathbf{q}$ which is sufficient for nonexistence of a time-invariant continuous state feedback law. Therefore, such a control law might exist.

5 Material Constraints

In practice, rarely do we need to control all degrees of freedom [10]. Steering only the directly controlled degrees of freedom is usually sufficient, provided that the rest are confined within specific limits. These limits naturally arise from material strength limitations and are written in the form:

$$\boldsymbol{\sigma} \leq \bar{\boldsymbol{\sigma}} \tag{11}$$

where $\boldsymbol{\sigma}$ is the stress tensor of the structure and $\bar{\boldsymbol{\sigma}}$ is the maximum admissible stress for the particular material and object. These constraints can be included in (9) through Kuhn-Tucker multipliers:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{B}\mathbf{u} + (\frac{\partial \mathbf{r}}{\partial \mathbf{q}})^T\boldsymbol{\mu}$$

$$\boldsymbol{\mu}^T\mathbf{r}(\mathbf{q}) = 0, \quad \boldsymbol{\mu} \geq 0$$

Note that in this case, the multipliers cannot be eliminated because they correspond to inequality conditions which do not reduce the dimension of the state space. Then the Kuhn-Tucker term can be included into the potential terms, yielding equations (2a)-(2b)

$$\mathbf{m}_{11}\ddot{\mathbf{q}}_1 + \mathbf{m}_{12}\ddot{\mathbf{q}}_2 + \mathbf{c}_1\dot{\mathbf{q}} + \mathbf{k}_1(\mathbf{q}, \boldsymbol{\mu}) = 0$$

$$\mathbf{m}_{21}\ddot{\mathbf{q}}_1 + \mathbf{m}_{22}\ddot{\mathbf{q}}_2 + \mathbf{c}_2\dot{\mathbf{q}} + \mathbf{k}_2(\mathbf{q}, \boldsymbol{\mu}) = \mathbf{b}\mathbf{u}$$

$$\boldsymbol{\mu}^T\mathbf{r}(\mathbf{q}) = 0, \quad \boldsymbol{\mu} \geq 0$$

When the stress conditions are satisfied, then $\boldsymbol{\mu}$ vanish and the equations describe the motion of an unconstrained system.

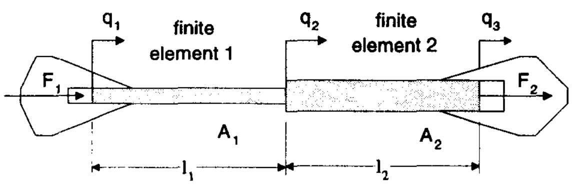

Figure 1: A deformable object under axial load

6 Examples

6.1 Rod under axial load

Consider a beam under axial load. The beam is divided into two finite elements (Figure 1). The system has three degrees of freedom, two of which are directly controlled. The element characteristic matrices are:

$$\mathbf{M} = \frac{\rho A \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{C} = \frac{\mu A \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{K} = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembling the element equations to form the complete equations and rearranging the terms to single out the actuated part from the unactuated, yields the dynamic equations

$$\begin{bmatrix} \frac{\rho(A_1\ell_1+A_2\ell_2)}{3} & \frac{\rho A_1 \ell_1}{6} & \frac{\rho A_2 \ell_2}{6} \\ \frac{\rho A_1 \ell_1}{6} & \frac{\rho A_1 \ell_1}{3} & 0 \\ \frac{\rho A_2 \ell_2}{6} & 0 & \frac{\rho A_2 \ell_2}{3} \end{bmatrix}\begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_1 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} \frac{\mu(A_1\ell_1+A_2\ell_2)}{3} & \frac{\mu A_1 \ell_1}{6} & \frac{\mu A_2 \ell_2}{6} \\ \frac{\mu A_1 \ell_1}{6} & \frac{\mu A_1 \ell_1}{3} & 0 \\ \frac{\mu A_2 \ell_2}{6} & 0 & \frac{\mu A_2 \ell_2}{3} \end{bmatrix}\begin{bmatrix} \dot{q}_2 \\ \dot{q}_1 \\ \dot{q}_3 \end{bmatrix} + \begin{bmatrix} \frac{A_1 E}{\ell_1} + \frac{A_2 E}{\ell_2} & -\frac{A_1 E}{\ell_1} & -\frac{A_2 E}{\ell_2} \\ -\frac{A_1 E}{\ell_1} & \frac{A_1 E}{\ell_1} & 0 \\ -\frac{A_2 E}{\ell_2} & 0 & \frac{A_2 E}{\ell_2} \end{bmatrix}\begin{bmatrix} q_2 \\ q_1 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1 \\ F_2 \end{bmatrix}$$

Where A_1, A_2 and ℓ_1, ℓ_2 are the elements cross sections and lengths, respectively. Applying the linearizing feedback (3) yields the familiar form (10a-10b)

$$\ddot{q}_1 = v_1$$
$$\ddot{q}_3 = v_2$$
$$\ddot{q}_2 = \mathbf{J}\begin{bmatrix} v_1 & v_2 \end{bmatrix}^T + R$$

where

$$\mathbf{J} = -\frac{1}{2(A_1\ell_1 + A_2\ell_2)}\begin{bmatrix} A_1\ell_1 & A_2\ell_2 \end{bmatrix}$$

$$R = -\frac{3}{\rho(A_1\ell_1 + A_2\ell_2)}\left[\frac{\mu}{3}(A_1\ell_1 + A_2\ell_2)\dot{q}_2 + \frac{\mu A_1 \ell_1}{6}\dot{q}_1 + \frac{\mu A_2 \ell_2}{6}\dot{q}_3 + E\left(\frac{A_1}{\ell_1} + \frac{A_2}{\ell_2}\right)q_2 - \frac{A_1 E}{\ell_1}q_1 - \frac{A_2 E}{\ell_2}q_3\right]$$

Using Proposition 1 we can conclude that the system is second-order nonholonomic. Indeed, in this case $r = 1$ and the matrix

$$L = \begin{bmatrix} \frac{\partial R}{\partial \dot{q}} \\ \frac{\partial R}{\partial q} + \frac{\partial R}{\partial \dot{q}_2}\cdot\frac{\partial R}{\partial \dot{q}} \end{bmatrix}\begin{bmatrix} J_1 & J_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has a determinant:

$$det[L] = -\frac{9(A_1\ell_1 + 2A_2\ell_2)E\mu A_2}{4\rho^2\ell_2(A_1\ell_1 + A_2\ell_2)^2} \neq 0$$

Therefore it is nonsingular and the system is second-order nonholonomic. This can be verified by taking the vector fields:

$$\tau_0 = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 & 0 & R & 0 & 1 \end{bmatrix}^T$$
$$\tau_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & J_1 & 0 & 0 \end{bmatrix}^T$$
$$\tau_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & J_2 & 1 & 0 \end{bmatrix}^T$$

and calculating a base for the accessibility algebra:

$$[\tau_0, \tau_1] = \begin{bmatrix} -1 & \frac{A_1\ell_1}{2(A_1\ell_1 + A_2\ell_2)} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$
$$[\tau_0, \tau_2] = \begin{bmatrix} 0 & \frac{A_1\ell_1}{2(A_1\ell_1 + A_2\ell_2)} & -1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$
$$ad^2_{\tau_0}\tau_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & G & 0 & 0 \end{bmatrix}^T$$
$$ad^2_{\tau_0}\tau_1 = \begin{bmatrix} 0 & -G & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

where $G = \frac{3A_1E(3A_1\ell_1\ell_2 + 2A_2\ell_2^2 + A_2\ell_1^2)}{2\rho(A_1\ell_1 + A_2\ell_2)^2\ell_1\ell_2} \neq 0$. Clearly the dimension of the accessibility algebra is $2n + 1$ and therefore the system is second-order nonholonomic by definition.

6.2 Beam under bending load

Consider a beam resisting bending moments applied on its plane (Figure 2). Here the finite element model is more complex. There are six degrees of freedom four of which, the linear and angular displacements at the grasp points, are actuated. Displacements q_3 and q_4 are not directly controlled. This system is also second-order nonholonomic. The matrix

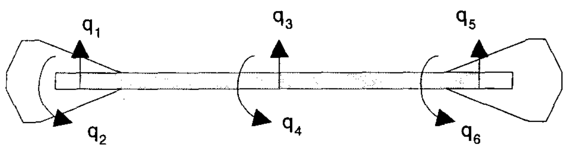

Figure 2: A deformable beam resisting bending

sequence has $\frac{2(n-m)}{m} = 1$ elements. Therefore we only need to calculate the matrix:

$$L = \begin{bmatrix} \frac{\partial \mathbf{R}}{\partial \dot{\mathbf{q}}} \\ \frac{\partial \mathbf{R}}{\partial \mathbf{q}} + \frac{\partial \dot{\mathbf{R}}}{\partial \dot{\mathbf{q}}_k} \cdot \frac{\partial \mathbf{R}}{\partial \dot{\mathbf{q}}} \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{I} \end{bmatrix} \quad k = 3, 4.$$

Its determinant turns out to be:

$$det[L] = \frac{4254529125}{562432} \frac{\mu^2 A^2 E^2 I^2}{m^4 \ell^4} \neq 0$$

indicating that the system is second-order nonholonomic. A direct calculation of a base for the accessibility algebra verifies the result.

7 Conclusion

A deformable object being manipulated can be discretized using finite elements and regarded as an underactuated mechanical system. In this framework many different kind of mechanical systems and manipulating tasks can be studied. We presented a sufficient condition for second-order nonholonomy, for an important class of deformable objects modeled with finite elements. This result permits to circumvent the usual procedure of calculating Lie brackets to establish a base for the associated Lie algebra, and answers the question of determining the kind of constraints imposed on the system in a straightforward algebraic way. Moreover, constraints associated with material strenght limitations have also been included.

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