Model Predictive Navigation for Position and Orientation Control of Nonholonomic Vehicles

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Abstract— **In this paper we consider a nonholonomic system in the form of a unicycle and steer it to the origin so that both position and orientation converge to zero while avoiding obstacles. We introduce an artificial reference field, propose a discontinuous control policy consisting of a receding horizon strategy and implement the resulting field-based controller in a way that theoretically guarantees for collision avoidance; convergence of both position and orientation can also be established. The analysis integrates an invariance principle for differential inclusions with model predictive control. In this approach there is no need for the terminal cost in receding horizon optimization to be a positive definite function.**

I. Introduction

Model predictive control allows the design of (sub)optimal control policies while avoiding solving directly the optimal control problem for an infinite time horizon. The method has been gaining popularity since it was first applied successfully in chemical process control, fueled by advances in both theoretical foundations [1] offering performance and stability guarantees, as well as computational hardware and software allowing for faster and more efficient computation. For linear control systems the field is already quite mature [2], but application to nonlinear systems remains a challenge [3]. A good overview of model predictive control and the different types applied can be found in [4].

This paper treats the problem of stabilizing a particular nonlinear system subject to state constraints. The system at hand is a unicycle, which serves as a model for a wide range of wheeled mobile robots, and our goal is to regulate both its *position as well as its and orientation* inside a workspace populated by obstacles. There is an extensive amount of research done in a) unconstrained nonholonomic stabilization, and b) control of *position* amongst obstacles [5], [6]; these are topics related but different from the goal of this paper, and thus only a selected subset of efforts along these lines is cited here. A notable alternative to the above methods, regulating vehicle position and orientation in constrained environments, is [7]; this is a multi-layered approach in which a (holonomic) path is first constructed linking the initial to final configurations, and then the path is locally approximated by feasible trajectories taking advantage of the system's small time local controllability properties.

Establishing stability and convergence to the origin is known to be nontrivial for nonholonomic systems, due to Brockett's necessary condition [8]. Brockett's conditions still apply to model predictive control solutions and thus asymptotic stability in the Lyapunov sense is still elusive. Invariance methods, however, ensure convergence without necessarily promising stability and are therefore appealing as a tool for analysis and design of nonholonomic controllers. Receding horizon optimization approaches that do not rely on terminal constraints [9], however, still require the use of a positive definite control Lyapunov function. With few notable exceptions [10], applying to unconstrained discrete-time nonlinear systems, there are not many options for ensuring convergence in continuous time using semidefinite terminal costs and without imposing terminal constraints or forcing switching near the desired equilibria.

Applying model predictive control strategies to nonholonomic control is not new. A receding horizon controller steers a unicycle in an environment without obstacles in [11]. Obstacle avoidance is treated in [12], however, the terminal cost used is required to be a positive definite functione.

In terms of collision-free nonholonomic navigation, solutions based on artificial (potential) fields are quite common. Examples include [13], with the caveat that it applies to fully actuated systems only, and [14] which uses artificial obstacles to shape the vehicle's final orientation. A navigation function [15] approach, with provable collision avoidance and convergence guarantees is found in [16]. One limitation of this method is that it may require a significant about of switching to ensure the decrease of the potential function. Other potential field-based approaches include [17], although it is not clear how it achieves nonholonomic stabilization using continuous feedback.

This paper suggests a discontinuous nonholonomic control strategy, that ensures obstacle avoidance and convergence of the unicycle's position *and* orientation. It couples a new type of artificial fields for navigation and obstacle avoidance with a receding horizon utilizing semi-definite terminal cost functions, and analyzes the convergence of the overall switching system in a differential inclusion framework using an invariance principle approach [18][1] (cf. [19]).

Section II that follows briefly presents the model predictive navigation scheme, some useful results from [18], and reviews the navigation function approach. Section III sets the main objective and formulates the problem, while Section IV presents the proposed control strategy. Section VI summarizes the paper.

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[1] Alternatives to [18] may be considered [20], [21], however these still require positive definiteness for the Lyapunov-like function.

II. PRELIMINARIES

A. Model Predictive Control

Consider a nonlinear system of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}) \ . \tag{1}$$

Denote $\boldsymbol{x} \in \mathbb{R}^n$ the vector of its generalized coordinates and $\boldsymbol{u} \in \mathbb{R}^m$ the control vector, with $n, m \in \mathbb{N}_+$. Given an initial condition $\boldsymbol{x}_0$ and a control input $\boldsymbol{u}$, the solution of (1) at time t under $\boldsymbol{u}(t)$, passing through $\boldsymbol{x}_0$ at $t = 0$ is expressed as

$$\boldsymbol{x}^{\boldsymbol{u}}(t; \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t f\big(\boldsymbol{x}^{\boldsymbol{u}}(\tau; \boldsymbol{x}_0), \boldsymbol{u}(\tau)\big)\, \mathrm{d}\tau \ . \tag{2}$$

For an initial state $\boldsymbol{x}$, we define the infinite horizon cost as the integral over the trajectories $\boldsymbol{x}^{\boldsymbol{u}}(t; \boldsymbol{x})$ of (1), given by

$$J(\boldsymbol{x}, \boldsymbol{u}(\cdot)) \triangleq \int_0^\infty q\big(\boldsymbol{x}^{\boldsymbol{u}}(\tau; \boldsymbol{x}), \boldsymbol{u}(\tau)\big)\, \mathrm{d}\tau \tag{3}$$

where $q(\cdot, \cdot)$ is a positive semi-definite function referred to as the *incremental cost*. The aim in the finite horizon optimization is to minimize the functional

$$J_T\big(\boldsymbol{x}, \boldsymbol{u}_T\big) \triangleq \int_0^T q\big(\boldsymbol{x}^{\boldsymbol{u}}(\tau; \boldsymbol{x}), \boldsymbol{u}_T(\tau)\big)\mathrm{d}\tau + V\big(\boldsymbol{x}^{\boldsymbol{u}}(T; \boldsymbol{x})\big) \tag{4}$$

that quantifies the cost of moving along a closed loop system trajectory $\boldsymbol{x}^{\boldsymbol{u}}(t; \boldsymbol{x})$ starting at $\boldsymbol{x}$, under the control law $\boldsymbol{u}_T(t, \boldsymbol{x})$, with $t \in [0, T]$. Here, $T > 0$ is the prediction horizon. The function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ in (4) is an approximation of the tail of the infinite horizon integral (3), truncated at T, denoted as the terminal cost.

The objective is to determine the optimal control law $\boldsymbol{u}_T^*$, which minimizes the finite horizon cost from $(\boldsymbol{x}, \boldsymbol{u}_T^*(\boldsymbol{x}; \cdot)) \triangleq \arg\min J_T\big(\boldsymbol{x}, \boldsymbol{u}_T(\cdot)\big)$, producing an optimal finite horizon trajectory $\big(\boldsymbol{x}_T^*(t; \boldsymbol{x}), \boldsymbol{u}_T^*(t; \boldsymbol{x})\big)$, for $t \in [0, T]$, with $\boldsymbol{x}_T^*$ denoting the optimal closed loop trajectory for that time interval. In a receding horizon strategy, $\boldsymbol{u}_T^*(t; \boldsymbol{x})$ is used for $t \in [0, \delta]$, with $\delta < T$, and then is recomputed with initial state $\boldsymbol{x}^{\boldsymbol{u}}\big(t; \boldsymbol{x}_T^*(\delta, \boldsymbol{x})\big)$. We denote δ the control horizon.

B. Invariance principle for differential inclusions

This section is a brief and incomplete review of relevant results, taken from [18]. Let the set-valued map $\boldsymbol{x} \mapsto F(\boldsymbol{x}) \subset \mathbb{R}^n$ with domain G to be upper semicontinuous with non-empty compact and convex values and $\boldsymbol{x}(t)$ to be a solution of the (Filippov) differential inclusion

$$\dot{\boldsymbol{x}} \in F(\boldsymbol{x}) \ . \tag{5}$$

Denote$S_{\boldsymbol{x}_0}$ the set of solutions of (5) satisfying $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

Definition 1: A solution $\boldsymbol{x}(t)$ of (5) is called *maximal* if it does not have a proper right extension which is also a solution.

Definition 2: A solution $\boldsymbol{x}(t)$ of (5) is *precompact* if it is maximal and the closure of its trajectory is compact.

For an interval $I \subset \mathbb{R}$ and $S \subset \mathbb{R}^N$, $AC(I; S)$ denotes the space of functions $I \to S$ that are absolutely continuous on compact subintervals of I. A function $\boldsymbol{x} \in AC(I; G)$ is said to be an X-arc if it satisfies the differential inclusion in (5) almost everywhere (note: $\boldsymbol{x}(t) \in G$).

Proposition 1: Every solution of (5) can be extended to a maximal solution. Moreover, if $\boldsymbol{x} \in AC([0, \alpha), G)$ is a precompact solution of (5), then $\alpha = \infty$.

Definition 3: Let $\boldsymbol{x} \in AC([0, \alpha), G)$ be a maximal solution of (5). A point $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is an *α-limit point* of $\boldsymbol{x}$ if there exists an increasing sequence $(t_n) \subset [0, \alpha)$ such that $t_n \to \alpha$ and $\boldsymbol{x}(t_n) \to \bar{\boldsymbol{x}}$ as $n \to \infty$. The set $\mathbb{A}(\boldsymbol{x})$ of all α-limit points of $\boldsymbol{x}$ is the *α-limit set* of $\boldsymbol{x}$.

Definition 4: Let $C \subset \mathbb{R}^n$ be non-empty. A function $\boldsymbol{x} \in AC([0, \alpha), G)$ is said to *approach* C, if $d_C(\boldsymbol{x}(t)) \to 0$ as $t \to \alpha$ where $d_C(\boldsymbol{y}) \triangleq \inf\{\|\boldsymbol{y} - c\|, \boldsymbol{c} \in C\}$.

Definition 5: Relative to (5), $S_{\boldsymbol{x}_0} \subset \mathbb{R}^n$ is said to be a *weakly invariant set* if for each $\boldsymbol{x}_0 \in S_{\boldsymbol{x}_0} \cup G$ there exists at least one maximal solution $\boldsymbol{x} \in AC([0, \alpha), G)$ of (5) with $\alpha = \infty$ and with trajectory $\boldsymbol{x}([0, \alpha))$ in $S_{\boldsymbol{x}_0}$.

Definition 6 ([22]): The generalized directional derivative $V^o(\boldsymbol{z}, \phi)$ of a locally Lipschitz function $V : G \to \mathbb{R}$ at $\boldsymbol{z}$ in the direction of ϕ is

$$V^o(\boldsymbol{z}; \phi) = \limsup_{\substack{y \to z \\ h \downarrow 0}} \frac{V(y + h\phi) - V(y)}{h} \ . \tag{6}$$

The map $(\boldsymbol{z}; \phi) \to V^o(\boldsymbol{z}; \phi)$ is upper semicontinuous and for each $\boldsymbol{z}$ the map $\phi \to V^o(\boldsymbol{z}; \phi)$ is Lipschitz continuous.

Theorem 1: Let $l : G \to \mathbb{R}$ be lower semicontinuous. Suppose that $U \subset G$ is non-empty and that $l(\boldsymbol{z}) \geq 0$ for all $\boldsymbol{z} \in U$. If $\boldsymbol{x}$ is a precompact solution of (5) with trajectory in U and $l \circ \boldsymbol{x} \in L^1(\mathbb{R}_+)$ then $\boldsymbol{x}(t)$ approaches the largest weakly invariant set in $\Sigma = \{\boldsymbol{z} \in \mathrm{cl}(U) \cap G : l(\boldsymbol{z}) \leq 0\}$.

Theorem 2: Let $V : G \to \mathbb{R}$ be locally Lipschitz. Define

$$\mathrm{u} : G \to \mathbb{R}, \ \boldsymbol{z} \to \mathrm{u}(\boldsymbol{z}) = \max\{V^o(\boldsymbol{z}; \phi) : \phi \in F(\boldsymbol{z})\} \ .$$

Suppose that $U \subset G$ is non-empty and that $\mathrm{u}(\boldsymbol{z}) \leq 0$ for all $\boldsymbol{z} \in U$. If $\boldsymbol{x}(t)$ is a precompact solution of (5) with trajectory in U, then for $c \in V(\mathrm{cl}(U) \cap G)$, $\boldsymbol{x}(t)$ approaches the largest weakly invariant set in $\Sigma \cap V^{-1}(c)$ where

$$\Sigma = \{\boldsymbol{z} \in \ cl(U) \cap G : \mathrm{u}(\boldsymbol{z}) \geq 0\} \ .$$

C. Navigation Functions

Consider a point robot moving in a planar environment populated by obstacles. Let $\boldsymbol{x} \triangleq (x, y)^T$ be the position of the robot and assume that $\boldsymbol{x} \in \mathbb{X} \triangleq \mathrm{cl}(B_R) \subset \mathbb{R}^2$, where $\mathrm{cl}(\cdot)$ denotes closure and B_R is an open ball of radius R containing the origin.[2] Assume the obstacles in the robot's workspace $\mathbb{X}$ are represented by open balls, $\mathbb{O}_i \in \mathbb{X}$, for $i \in \{1, \ldots, m\}$. These spheres are isolated, that is $\mathbb{O}_i \cap \mathbb{O}_j = \emptyset$ for every $i \neq j$ with $i, j \in \{0, \ldots, m\}$, where index 0 denotes the workspace boundary. This is defined as an obstacle, expressed as the complement of $\mathbb{X}$ in $\mathbb{R}^2$: $\mathbb{O}_0 \triangleq \mathbb{R}^2 \setminus \mathbb{X}$.

A navigation function [15], [23] for the considered robot can be defined as a map $\varphi : \mathbb{X} \setminus \bigcup_{i=0}^m \mathbb{O}_i \to [0, 1]$ which

[2]Note that star-shaped region can be diffeomorphically transformed into a sphere [15].

1) is smooth (or at least twice differentiable),
2) has a unique minimum at a single point $\boldsymbol{x}_d \in \mathbb{X} \setminus \bigcup_{i=0}^{m} \mathbb{O}_i$,
3) is uniformly maximal on the boundary of $\mathbb{X} \setminus \bigcup_{i=0}^{m} \mathbb{O}_i$,
4) is a Morse function.

Define the function

$$\varphi(\boldsymbol{x}) = \frac{\|\boldsymbol{x} - \boldsymbol{x}_d\|^2}{\left(\|\boldsymbol{x} - \boldsymbol{x}_d\|^{2\kappa} + \beta(\boldsymbol{x})\right)^{1/\kappa}} \ , \tag{7}$$

where $\beta = \prod \beta_i$ and $\forall i \in \{1, \ldots, m\}, \beta_i = \|\boldsymbol{x} - \boldsymbol{x}_i\| - r_i$. Each β_i represents a spherical obstacle of radius r_i centered at $\boldsymbol{x}_i \in \mathbb{X}$. Then, for a sufficiently large value of the parameter $\kappa > 0$, (7) is a navigation function [24].

III. Main Objective and Problem Statement

In this paper we consider a point robot with the kinematics of a unicycle

$$\begin{aligned} \dot{x} &= v \cos\theta \\ \dot{y} &= v \sin\theta \\ \dot{\theta} &= \omega \end{aligned} \tag{8}$$

where $\boldsymbol{x} = (x, y)^T \in \mathbb{X} \subset \mathbb{R}^2$ is the position vector and $u = (v, w)^T \in \mathbb{R}^2$ is the control input.

We want to find a static state feedback controller, which with minimal switching enables the unicycle to converge to the origin with zero orientation, i.e., $\boldsymbol{x}(t) \to \mathbf{0}$ *and* $\theta \to 0$ as $t \to \infty$, while staying away of certain areas of the field (obstacles and attraction regions of saddle points of φ). We will formally define this admissible set of states as the *operational workspace* in Section IV.

IV. Technical Approach

We propose a discontinuous controller which induces two component dynamics for the resulting switching system: a) a model predictive control mode, and b) an *artificial reference field* alignment mode. The artificial reference field is a two dimensional vector field on $\mathbb{X}$ constructed in a way that its integral lines pass through the origin with a desired orientation. This vector field will be used as a reference velocity generator for the system. Because of the fact that all integral lines of that field pass through the origin with the same orientation, we call this field, *dipolar*.

A. Properties of the dipolar field

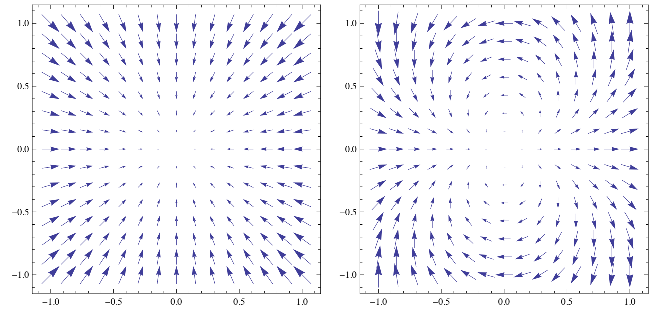
Consider the map $\Gamma(z) : \mathbb{R}^2 \to \mathbb{R}^2$

$$\Gamma(z) = \begin{pmatrix} -\cos z & \sin z \\ -\sin z & -\cos z \end{pmatrix}$$

where

$$\begin{aligned} z &\triangleq s_d(x, y; a) \arctan 2(y, x) + \pi \mathrm{sign}(y)(1 - s_d(x, y; a)) \\ s_d(x, y; a) &\triangleq \exp\left(-\frac{a}{(1 - \varphi(x, y))^2} + a\right) \ , \end{aligned}$$

and a is a positive parameter. This map is a discontinuous, nonsingular map that rotates a vector at a given point (x, y) by an angle equal to the vector from the origin to (x, y).



(a) Stable proper node. (b) Rotated field via Γ.

Fig. 1. The effect of Γ on a convergent vector field. The field in 1(b) was produced by mapping the field in 1(a) through a map Γ in which s_d was set identically to one.

Application of this map to the vector field defined by the negated gradient of (7) (i.e., $-\nabla_{\boldsymbol{x}} \varphi(\boldsymbol{x})$) produces a *dipolar potential field* denoted $DF(\boldsymbol{x})$.

Definition 7: We define the dipolar potential field as the image of the negated gradient of a navigation function under the transformation $\Gamma(\boldsymbol{x})$

$$DF(\boldsymbol{x}) \triangleq -\Gamma \ \nabla_{\boldsymbol{x}} \varphi(\boldsymbol{x}) \tag{9}$$

A number of properties can be shown for $DF(\boldsymbol{x})$.

Lemma 1: All integral manifolds of $-\Gamma(x, y)^T$ are closed lines that contain the origin, and the latter is the only singular point of the vector field. Furthermore, as the integral lines approach the origin, the direction of the tangent vector converges to zero.

Proof: Let us consider the two fields $-\Gamma(x, y)^T$ and $(x^2 - y^2, 2xy)^T$. The cross product of the two vector fields when embedded in $\mathbb{R}^3$ produces a third, which has as its third component the expression $-y \cos(\arctan 2(y, x)) + x \sin(\arctan 2(y, x)) \equiv 0$. This means that at any point $(x, y, 0)$, the directions of vector fields $-\Gamma(x, y)^T$ and $(x^2 - y^2, 2xy)^T$, coincide. The latter field, $(x^2 - y^2, 2xy)^T$, is known in literature [25] to produce closed orbits which are circles tangent to each other at the origin.

The fact that the origin is the only singular point follows from Γ being nonsingular, and $-(x, y)$ has the origin as a stable node. Therefore, as the flow of any one of these vector fields moves along these orbits, it eventually encounters the origin. On the other hand, the argument that the direction of the vector fields, given by $2 \arctan 2(y, x)$ converges to zero as the integral line approaches the origin is found in [26]. ■

Lemma 2: The field $-\Gamma \ \nabla\varphi$ preserves the collision avoidance properties of the field $-\nabla\varphi$ that has been generated by the navigation function φ.

Proof: On the boundary of the free workspace, $\partial \bigcup_{i=1}^{m} \mathbb{O}_i(\boldsymbol{x}) = 1$ and therefore $s_d(x, y; a) = 0$. When this happens, $z = \pi \ \mathrm{sign}(y) = \pm\pi$ which means that $\Gamma(z) = I_2$, where I_2 is the identity matrix on $\mathbb{R}^2$. If φ is a navigation function, then its negated gradient points away from $\partial \bigcup_{i=1}^{m} \mathbb{O}_i$. On the workspace boundary, the transformed field coincides with the navigation function field, and thus inherits its collision avoidance properties. ■

Lemma 3: Transformation Γ introduces stationary points with empty attraction regions.

Proof: The Γ function exhibits discontinuity on the $y = 0$ line, because of the sign function it involves. The positive x semi-axis is particularly problematic, because this is where the transformed field $-\Gamma\ \nabla\varphi$ will be directly opposed that of $-\nabla\varphi$ (Fig. 1); somewhere on this line the vector field will have to reverse direction, and this is exactly where the stationary point appears.

For any small neighborhood of $(x, y) = (0, 0)$ and any $\epsilon > 0$, there are points in this neighborhood for which there is a sufficiently small a such that $s_d(x, y; a) < \epsilon$. This essentially ensures that the transition, or "blending," of the dipolar field of Fig. 1(b) with that of $-\nabla\varphi$ occurs within that neighborhood. Note that as $(x, y) \to (0, 0)$, since $\prod_{i=1}^{m} \beta_i$ converges to a constant and $\Gamma(x, y) \to 0$, we have $\varphi(x, y) \propto \|\boldsymbol{x}\|^2$. Naturally, $\langle -\nabla\varphi, -\boldsymbol{x}\rangle > 0$ there,[3] and locally, the negated gradient field of φ resembles that of the stable node shown in Fig. 1(a).

Along the x-axis where $y = 0$, $-\boldsymbol{x}$ is tangent to the x-axis and along each side of the axis the vector field of $-\boldsymbol{x}$ points toward the axis. In the same area, where y is very small and x is positive, $\Gamma(\boldsymbol{x})$ rotates $-\boldsymbol{x}$ by more than $\frac{\pi}{2}$, so that $\langle \Gamma(\boldsymbol{x})(-\boldsymbol{x}), -\boldsymbol{x}\rangle < 0$. What this implies is that the Filippov solutions [27] of $-\Gamma\nabla\varphi$ that start from the discontinuity line when $x > 0$ diverge from it. In contrast, the Filippov solutions that start at $y = 0$ when $x < 0$ form a sliding mode and "slide" along the axis as they approach the origin —these are not problematic because they converge with the desired orientation. Thus, along the discontinuity surface the only sliding motion that exists converges to the origin, whereas no solution converging to the stationary point at the positive x semi-axis exists. ■

Given $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$, their *Minkowski sum* is $\mathcal{A} \oplus \mathcal{B} \triangleq \bigcup_{b\in\mathcal{B}}(\mathcal{A} + b)$, while their *Minkowski difference* is $\mathcal{A} \ominus \mathcal{B} \triangleq \bigcap_{b\in\mathcal{B}}(\mathcal{A} - b)$. Define the following sets:

- $\bigcup_{\mathrm{i}=1}^{l} \mathbb{S}_i$ is the set that contains all stationary (singular) points of DF.
- $\mathbb{L}$ is the set containing all the integral lines of the dipolar field that pass through a critical point of φ.
- $W_\epsilon = \{\bigcup_{\mathrm{i}=1}^{m} \mathbb{O}_i \cup \bigcup_{\mathrm{i}=1}^{l} \mathbb{S}_i \cup \mathbb{L}\} \oplus B_\epsilon$ for some $\epsilon > 0$ represents an undesirable region for the position of the system. Note that B_ϵ denotes an open ball of radius ϵ. Initial states too close to the critical points of φ and their attraction regions will turn out to be problematic for establishing convergence due to the vanishing gradient.
- $G = \{\boldsymbol{x} \in \mathbb{X}\backslash W_\epsilon\} \times \mathsf{S}^1$ is the operational workspace,[4] that is, the set of admissible states for the system. Note that G is compact by its definition.

Corollary 1: The integral lines of DF on the boundary of G point inwards.

Proof: By Lemma 3, it follows that saddle points of φ have regions of attraction of measure zero. Then, any small neighborhood of this regions cannot be invariant. At the boundary of some $\mathbb{O}_i$, on the other hand, Lemma 2 suggests that DF points away from $\mathbb{O}_i$. Then, by continuity there exists ϵ-neighborhood for which DF points inside G. ■

Lemma 4: All integral lines of $-\Gamma(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})$ converge to $\boldsymbol{x} = 0$ asymptotically along the negative x axis, and their common derivative at $\boldsymbol{x} = 0$ is aligned with the x axis.

Proof: The integral lines of DF are bounded in a compact set $\mathrm{cl}(\mathbb{X} \setminus \bigcup_{i=1}^{m} \mathbb{O}_i)$, so there is a limit set inside the compact set where the trajectories converge. The critical points of $-\nabla\varphi$ other than $\boldsymbol{x} = 0$ do not change nature under the transformation, because Γ is continuous around any one of these critical points, and φ is Morse. Thus all critical points other than the destination $\boldsymbol{x} = 0$ remain unstable, with an attraction region of measure zero. Unless there are additional attractors, the origin must be the only attractive component of the limit set.

In principle, the transformation induced by Γ may introduce a limit cycle. For a sufficiently small a, Γ is bound to apply only around $\boldsymbol{x} = 0$, so if there is a limit cycle the equilibrium encircled must be $\boldsymbol{x} = 0$. This implies that that hypothetical limit cycle must intersect with the x-axis. However, the proof of Lemma 3 states that Γ makes the positive x-axis around the origin, repulsive. Therefore, it is not possible for trajectories to cross it on their way to the hypothetical attractive limit cycle. By contradiction, the possibility of limit cycles existing around the origin is excluded, leaving the origin as the only attractive component of the positive limit set in $\mathrm{cl}(\mathbb{X} \setminus \bigcup_{i=1}^{m} \mathbb{O}_i)$.

To see why the integral lines of the vector field $F(\boldsymbol{x})$ near the origin (Fig. 1(b)) approach $\boldsymbol{x} = 0$ with zero slope, note first that these lines cannot reach the origin from the right half plane; this is because the positive x-axis is rendered repulsive. On the left half plane, near the origin and close to the x-axis, $\Gamma(\boldsymbol{x})$ converges to the identity matrix; along the y-axis the $-\Gamma\nabla\varphi(\boldsymbol{x})$ tends to $(\pm y, 0)^\top$, with the sign depending on the side of the x axis the limit is evaluated on. The only possible direction of approach to the origin for the integral lines of $-\Gamma\nabla\varphi(\boldsymbol{x})$ is along the negative x-axis, which suggests that the slope of the vector field $-\Gamma\nabla\varphi$ needs to tend to zero as $x \to 0^-$. ■

B. Convergence

The closed loop system (8) is a switched system

$$\dot{\boldsymbol{z}} = f_\sigma(\boldsymbol{z}, u) \tag{10}$$

with $\boldsymbol{z} = (\boldsymbol{x}, \theta)$ and Ω denoting its space of solutions. Assume that all initial conditions for $\boldsymbol{x}$ belong in G. In (10)

- $\sigma \in \{1, 2\}$ is the discrete system mode indexing the component continuous dynamics.
- With ω_{RH} being a rotational velocity input to be determined by a receding horizon optimization algorithm,

$$u = (v, \omega) = \begin{cases} (k_v \tanh(x^2 + y^2), \omega_{\mathsf{RH}}) & \text{if } \sigma = 1 \\ (0, -k_\omega(\theta - \theta_d)) & \text{if } \sigma = 2 \end{cases}$$

where k_v and $k_\omega \geq 0.5$ are positive constants and $\theta_d = \arctan 2(DF_y, DF_x)$, with $(DF_x, DF_y)^T = DF(\boldsymbol{x})$.

[3] $\langle\cdot\rangle$ denotes inner product.

[4] S^k denotes a k-dimensional circle.

- The system operates in mode $\sigma = 2$ iff

$$\boldsymbol{x} \in \{\mathbf{x} \in \partial W_\epsilon : \langle \widehat{DF}, \eta \rangle \leq \langle \widehat{DF}, (\cos\theta, \sin\theta) \rangle\}$$

where $\eta \in \mathbb{R}^2$ is the vector tangent to the boundary of the undesirable region for the position, ∂W_ϵ.

Define the incremental and terminal costs as

$$q(\boldsymbol{x};\theta) \triangleq k(\theta - \theta_d)^2 = V(\boldsymbol{x}) \quad (11)$$

with k a positive constant.

Proposition 2: There exists an ω_{RH} in (10) such that

$$\dot{V}(\boldsymbol{x};u) + q(\boldsymbol{x};\theta) \leq 0 \quad (12)$$

holds $\forall\, t \geq 0$.

Proof: We have $\dot{V} = 2k(\theta - \theta_d)(\omega - \dot{\theta}_d)$. Thus (12) $\iff 2k(\theta - \theta_d)(\omega - \dot{\theta}_d) \leq -k(\theta - \theta_d)^2$. If $\theta = \theta_d$ then (12) trivially holds. For $\theta \neq \theta_d$, $\dot{V} \leq -q$ is equivalent to

$$|\theta - \theta_d| \leq 2|\omega - \dot{\theta}_d|. \quad (13)$$

In addition, $\dot{\theta}_d = v(\frac{\partial \theta_d}{\partial x}\cos\theta + \frac{\partial \theta_d}{\partial y}\sin\theta)$. For a bounded v, there is always an ω to satisfy (13). ■

Proposition 3: When $\sigma = 2$, the unicycle's orientation aligns with the integral lines of DF exponentially fast.

Proof: By Proposition 2 we have $\dot{V} \leq -q \implies \dot{q} \leq -q$. The comparison lemma [28] implies

$$\begin{aligned} &0 \leq q(t) \leq q(0)e^{-t} \iff \\ &0 \leq (\theta(t) - \theta_d(\boldsymbol{x}(t)))^2 \leq (\theta_0 - \theta_d(\boldsymbol{x}_0))^2 e^{-t}\ . \end{aligned} \quad (14)$$

Then, $\theta(t) \to \theta_d(\boldsymbol{x}(t))$ exponentially fast. ■

Proposition 4: Let $E = \{\boldsymbol{x} \in \Omega : \dot{V} = 0\}$. The largest invariant set in E is $\Sigma = \{\boldsymbol{x} \in \Omega : q(\boldsymbol{x};\theta) = 0\}$.

Proof: Let $\sigma = 1$. By its definition, the terminal cost V is a continuously differentiable function and since (12) can be assumed to hold $\forall t \geq 0$ we have that for all time $\dot{V} \leq -q \leq 0$. There are two cases for which $\dot{V} = 0$, namely $\{\theta - \theta_d = 0\}$ and $\{\omega - \dot{\theta}_d = 0\}$. Suppose $\{\theta - \theta_d = 0\}$; then necessarily $\omega = \dot{\theta}_d$. Now suppose that $\omega = \dot{\theta}_d$, with $\theta \neq \theta_d$. For the set to be invariant, we need $\omega - \dot{\theta}_d = \dot{\omega} - \ddot{\theta}_d = \ddot{\omega} - \dddot{\theta}_d = \cdots = 0$. But when $\omega = \dot{\theta}_d$ it follows that there exists some $c \triangleq \theta - \theta_d \neq 0$ implying $q = kc^2 > 0\ \forall\, t \geq 0$. Then $\dot{V} < 0\ \forall t \geq 0$, and this implies that there exists finite $T > 0$ such that $V(T) < 0$ which is a contradiction. As a result, $\theta - \theta_d = 0$ too and then $\theta - \theta_d = 0 \iff q(\boldsymbol{x};\theta) = 0$ so that $\Sigma = \{\boldsymbol{x} \in \Omega : q(\boldsymbol{x};\theta) = 0\}$ is the largest invariant set in E where $\dot{V} = 0$.

Now let $\sigma = 2$. Then $\omega = -k_\omega(\theta - \theta_d)$ aligns the unicycle's orientation with that of the dipolar field exponentially fast while $\boldsymbol{x}$ remains constant. There is a finite time at which the system will switch back to $\sigma = 1$. During the time that $\sigma = 2$, note that $\dot{V} = -2kk_\omega(\theta - \theta_d)^2 \leq -k(\theta - \theta_d)^2 = q(\boldsymbol{x}, \theta)$; thus the discussion of the previous paragraph applies.

When evaluated on the boundary ∂W_ϵ, the component vector fields of (10) on each side do not point to the other side. Thus, [29] the boundary is not a sliding surface and therefore not invariant. Then, since there is no invariant set when switching occurs, any invariant set must be inside the complement of $\{\mathbf{x} \in \partial W_\epsilon : \langle \widehat{DF}, \eta \rangle \leq \langle \widehat{DF}, (\cos\theta, \sin\theta) \rangle\}$ in G. ■

Theorem 3: The switched system (10) converges to the set $\Sigma = \{\boldsymbol{x} \in \Omega : q(\boldsymbol{x};\theta) = 0\}$ under a receding horizon strategy. Moreover, convergence to $\Sigma = \{\boldsymbol{x} \in \Omega : q(\boldsymbol{x};\theta) = 0\}$ implies convergence to the origin.

Proof: Let ω_{RH} be selected as a result of a receding optimization strategy that abides to (13), and recall that for finite dimensional spaces, the Filippov inclusion of the closed loop (10) is given as [20]

$$F(\mathbf{z}) = \mathrm{cl}\left(\mathrm{co}\left\{\lim_{\boldsymbol{z}_i \to \boldsymbol{z}} f_\sigma(\boldsymbol{z}) \mid \boldsymbol{x}_i \notin \partial W_\epsilon\right\}\right)$$

where $\mathrm{co}\{\cdot\}$ denotes convex hull. Then, when the system switches between modes, and given that for both modes $\dot{V} \leq -q(\boldsymbol{x};\theta)$, for $\phi \in F(\mathbf{z})$, $V^o(\mathbf{z};\phi)$ must necessarily satisfy $V^o(\boldsymbol{z};\phi) \leq -q(\boldsymbol{x};\theta)$.

The fact that ∂W_ϵ is not a sliding surface and that the component vector fields there point outside W_ϵ practically means that a solution starting in the closure of G will never leave this set, so $\Omega \subseteq G$. In addition, Ω is non-empty, compact (since G is also compact) and positively invariant. A solution $\boldsymbol{x}(t) \in \Omega$ is maximal, since it stays in G for all times and as Ω is compact, it follows that the solution $\boldsymbol{x}(t)$ is precompact.

Moreover, $V : G \to \mathbb{R}$ is locally Lipschitz. Define $\zeta : G \to \mathbb{R}, \boldsymbol{x} \to \zeta(\boldsymbol{x}) = \max\{V^o(\boldsymbol{z};\phi) : \phi \in F(\boldsymbol{z})\}$. We have proven that the stability condition (12) holds at all times, in both modes and during switching so, $\zeta(\boldsymbol{x}) \leq -q(\boldsymbol{x};\theta) \leq 0$ for all $\boldsymbol{x} \in \Omega$. In addition, define the set $\mathbb{S} = \{\boldsymbol{x} \in \mathrm{cl}(\Omega) \cap G : \zeta(\boldsymbol{x}) \geq 0\}$. Then all the hypotheses of Theorem 2 are met and so, for some constant $c \in V(\mathrm{cl}(\Omega) \cap G)$, the solution $\boldsymbol{x}(t)$ approaches the largest weakly invariant set in $\{\mathbb{S} \cap V^{-1}(c)\} = E = \{\boldsymbol{x} \in \Omega : \dot{V} = 0\}$. By Proposition 4 the largest weakly invariant set in E is the set $\Sigma = \{\boldsymbol{x} \in G : q(\boldsymbol{x};\theta) = 0\}$. Thus, a receding horizon strategy will make the unicycle converge to the set $\Sigma = \{\boldsymbol{x} \in \Omega : q(\boldsymbol{x};\theta) = 0\}$.

By the definition of q it follows that $q(\boldsymbol{x};\theta) = 0 \iff (\theta - \theta_d) = 0$. If $(\theta - \theta_d) = 0 \iff \theta = \theta_d$, the unicycle's orientation is aligned with a dipolar field's integral line and moves along it. By Lemma 4, all the integral lines of the dipolar field converge to $\mathbf{x} = 0$, with $\theta = 0$. ■

V. Discussion

A key feature of the work presented in this paper is the control of the orientation. This is crucial when we want to perform tasks that combine obstacle avoidance with other tasks that require a specific final orientation. Moreover, navigation function-based methods are typically tuned by trial and error, and if parameters are not set appropriately, convergence guarantees may be lost—a receding horizon approach allows the system to avoid myopic navigation decisions and alleviates the impact of inappropriate tuning. Still, the real-time, state feedback character of the control loop is preserved.

Dipolar vector fields have limitations. One issue is that small perturbations along the axis orthogonal to the direction of desired orientation in the neighborhood of the destination generate large orientation errors $\theta - \theta_d$ which may in turn produce high amplitude input oscillations. In that respect, a model predictive loop introduces dwell time between control updates and alleviates this problem. Problems of this nature manifest themselves in real implementations—not in numerical simulations. In preliminary tests we performed with skid-steering mobile robots, which have inherently inaccurate orientation control, we observed the generation of both orientation, as well as position errors as the platform approached its desired posture. The latter are due to the discrete-time nature of (piece-wise constant) control input update which tends to force the system to overshoot its desired position. The nature of the dipolar reference field couples position and orientation, and thus complicates the treatment of such errors.

In part, some of these problems are inherent in nonholonomic stabilization. Along the lines of the particular approach, however, we have obtained better results in terms of residual orientation errors by "dilating" the vector field in the x direction, and terminating the control action once the system has reached a sufficiently small neighborhood of the desired posture.

VI. Conclusions

A specially constructed vector field, which cannot be derived as a gradient of a potential function, is constructed. All of the integral lines of the field pass through the origin, and thus the field serves as a velocity reference for a unicycle. A discontinuous control law, involving a receding horizon scheme in one of its modes steers the unicycle along the flow lines of the field and enables the system to converge to the origin with a specific orientation. The approach to convergence analysis introduces a novel integration of Ryan's invariance principle for differential inclusions with receding horizon control, that guarantees convergence without relying on the positive definiteness properties of the terminal cost.

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