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Erratum to: Constrained Decision-making for Low-count Radiation Detection by Mobile Sensors

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The main reason for this erratum is that some specific proof techniques used can have unintended causality implications, related to the interpretation of differential equations of the form $\dot{x}(t) = g(x, \int_{t_0}^{t_f} f(x(t))dt)$. Both intermediate (Lemma 3) and end results (Proposition 1) are correct as originally stated; yet, we want to present an alternative proof pathway that circumvents the causality issue. For completeness and self-sufficiency of this document, we chose to include the whole revised proofs as opposed to introducing isolated, point modifications. Some derivations are listed in slightly more detail.

Typos:

a symbol missing in the proof of Lemma 1, which is also stated here in its revised entity;

In the proof of the Lemma 4: symbol ϵ should be ξ;

In the proof of the Theorem 2:

$$\ddot{\varphi} = -c\frac{\xi + \frac{1}{2}\|\nabla_{x_i}\varphi_i\|}{(\|\nabla_{x_i}\varphi_i\|+\xi)^2}\frac{\mathrm{d}\|\nabla_{x_i}\varphi_i\|^2}{\mathrm{d}t} \quad .$$

Lemma 1 *Fix $\boldsymbol{u}$. F_{FA} is strictly increasing with p.*

Proof Write $\frac{\partial F_{FA}}{\partial p} = \sum_{i=1}^{k_s}\int_0^T p\,\mu_i^p\,(\log\mu_i)^2 b_i\,\mathrm{d}t$, and note it is strictly positive since $\mu_i > 1, p \in (0,1)$. □

Lemma 3 *Let $\boldsymbol{\mu}$ be perturbed by $\epsilon\delta(t-t_1)$ on μ_i for some $t_1 \in (0,T]$, yielding $\tilde{\boldsymbol{\mu}}$. Consider $p(t,\tilde{\boldsymbol{\mu}}) = \phi(\boldsymbol{\mu}) + \sum_{i=1}^{k_s}\frac{\partial p}{\partial\mu_i}\Big|_t(\tilde{\mu}_i - \mu_i)$. For all $t_1 \in (0,T]$, if $\frac{\partial p}{\partial\mu_i}\Big|_{t_1} = -\frac{b_i\phi(\boldsymbol{\mu})\,\mu_i^{\phi(\boldsymbol{\mu})-1}\log\mu_i}{\sum_{j=1}^{k_s}\mu_j^{\phi(\boldsymbol{\mu})}\,\log^2\mu_j b_j}\Big|_{t_1} < 0$, then $F_{FA}(\tilde{\boldsymbol{\mu}}, \tilde{p} = p(t,\tilde{\boldsymbol{\mu}})) = -\log\alpha$.*

Proof Consider first a needle perturbation of the form $\epsilon\delta(t-t_1)$ on coordinate i of $\boldsymbol{\mu}$, yielding a perturbed $\tilde{\boldsymbol{\mu}}$ with component $\mu_i(t) + \epsilon\delta(t-t_1)$; here, $\delta(t-t_1)$ is the Dirac function offset at t_1 and $\epsilon > 0$ is a small parameter. Using Taylor expansion on the integrand of $F_{FA}(\boldsymbol{\mu}, p = \phi(\boldsymbol{\mu}))$ we find

$$F_{FA}(\tilde{\boldsymbol{\mu}}, p) \approx F_{FA}(\boldsymbol{\mu}, p) + b_i\epsilon p^2\mu_i(t_1)^{p-1}\log\mu_i(t_1) \quad ,$$

from which the first order variation in $F_{FA}(\boldsymbol{\mu}, p)$ due to $\epsilon\delta(t-t_1)$ in μ_i is obtained

$$\begin{aligned}&F_{FA}(\tilde{\boldsymbol{\mu}}, \tilde{p}) - F_{FA}(\boldsymbol{\mu}, p)\\ &= \left[b_i\epsilon p^2{\mu_i}^{p-1}\log\mu_i + \left[\sum_{j=1}^{k_s} p\,\mu_j^p\,\log^2\mu_j b_j\right]\frac{\partial p}{\partial\mu_i}\epsilon\right]\Bigg|_{t_1}\\ &\qquad + o(\epsilon) \quad ,\end{aligned}$$

which is zero because F_{FA} is constrained to $-\log\alpha$.

$$\begin{aligned}\frac{\partial p}{\partial\mu_i}\bigg|_{t_1} &= -\lim_{\epsilon\to 0}\frac{b_i\epsilon\phi(\boldsymbol{\mu})\,\mu_i^{\phi(\boldsymbol{\mu})-1}\log\mu_i + o(\epsilon)}{\sum_{j=1}^{k_s}\mu_j^{\phi(\boldsymbol{\mu})}\,\log^2\mu_j b_j\epsilon}\bigg|_{t_1}\\ &= -\frac{b_i\phi(\boldsymbol{\mu})\,\mu_i^{\phi(\boldsymbol{\mu})-1}\log\mu_i}{\sum_{j=1}^{k_s}\mu_j^{\phi(\boldsymbol{\mu})}\,\log^2\mu_j b_j}\bigg|_{t_1} < 0.\end{aligned}$$

and the proof is completed. □

Proposition 1 *The solution for sensor $i \in \{1,\ldots,k_s\}$ to the optimal control problem (5)–(7) within the feasible set $\mathcal{U} = \{\boldsymbol{u} \in \mathbb{R}^{3k_s} : \|u_i\| \le u_{\max}\}$ is*

$$u_i = \begin{cases}\frac{x_t - x_i}{\|x_t - x_i\|}u_{\max} & x_i \neq x_t\\ \dot{x}_t & x_i = x_t\end{cases} .$$

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Proof Given (8), the cost functional is written

$$J_{\mathrm{PM}} = \sum_{i=1}^{k_s} \int_0^T (\mu_i^p \log \mu_i - \mu_i + 1) b_i \, \mathrm{d}t \ .$$

Since J_{PM} is always finite, by Fubini's theorem,

$$J_{\mathrm{PM}} = \int_0^T \sum_{i=1}^{k_s} (\mu_i^p \log \mu_i - \mu_i + 1) b_i \, \mathrm{d}t \ .$$

Along the optimal trajectory $(\boldsymbol{\mu}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*)$, we define:

$$p = p^* + \sum_{i=1}^{k_s} \frac{\partial p}{\partial \mu_i} (\mu_i - \mu_i^*)$$

where p^* can be any constant value in $(0,1)$, which may include $\phi(\mu^*)$. Thus the Hamiltonian can be written as:

$$\begin{aligned} &H(\boldsymbol{\mu}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*) \\ &\quad = \sum_{i=1}^{k_s} \lambda_i^* \dot{\mu}_i(u_i^*) - \sum_{i=1}^{k_s} (\mu_i^{*p} \log \mu_i^* - \mu_i^* + 1) b_i \end{aligned} \quad (9)$$

and dynamics of costate λ_i is written as

$$\begin{aligned} \dot{\lambda}_i^* &= -\frac{\partial H}{\partial \mu_i}(\boldsymbol{\mu}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*) \\ &= \sum_{j=1, j\neq i}^{k_s} (\mu_j^{*p^*} \log^2 \mu_j^* \frac{\partial p}{\partial \mu_i}) b_j + \Big[\mu_i^{*p^*} \log^2 \mu_i^* \frac{\partial p}{\partial \mu_i} \\ &\quad + p^* \mu_i^{*p^*-1} \log \mu_i^* + \mu_i^{*p^*-1} - 1 \Big] b_i \\ &= \sum_{j=1}^{k_s} (\mu_j^{*p^*} \log^2 \mu_j^* b_j) \left[- \frac{b_i p^* \ \mu_i^{*p^*-1} \log \mu_i^*}{\sum_{j=1}^{k_s} \mu_j^{*p^*} \ \log^2 \mu_j^* b_j} \right] + \\ &\quad \Big[p^* \mu_i^{*p^*-1} \log \mu_i^* + \mu_i^{*p^*-1} - 1 \Big] b_i \\ &= [\mu_i^{*p^*-1} - 1)] b_i \end{aligned}$$

Since $\mu_i^* > 1$ and $p^* \in (0,1)$, we have $0 < \mu_i^{*p^*-1} < 1$, and therefore

$$\dot{\lambda}_i^* < 0 \quad (10)$$

for all $t \in (0,T]$.

Now since $\mu_i^*(T)$ can take any value in $(1, 1+\frac{a}{2b_i}]$, there are two mutually exclusive and exhaustive cases: either $\mu_i^*(T) \in (1, 1+\frac{a}{2b_i})$, or $\mu_i^*(T) = \mu_{i_{max}} = 1 + \frac{a}{2b_i}$. If $\mu_i^*(T) \in (1, 1+\frac{a}{2b_i})$, the transversality condition requires $\lambda_i^*(T) = 0$. Thus, given (10), it is $\lambda_i^*(t) > 0$ $\forall t \in (0,T]$. In light of this, and given (7) the Hamiltonian maximization condition $H(\boldsymbol{\mu}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*) = \max_{\boldsymbol{u} \in \mathcal{U}} H(\boldsymbol{\mu}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ applied on (9) requires that

$$u_i^* = \frac{x_t - x_i}{\|x_t - x_i\|} u_{\max} \ , \quad (11)$$

that is, it suggests the maximal control effort to close the distance between sensor and source as close as possible. Using such a controller, eventually (given big enough T) it will be $\mu_i^*(T) = \mu_{i_{max}}$. At this point, the second case is in effect. Denote T_s the switching time. Now $t \in [T_s, T]$ with boundary condition $\mu_i^*(T_s) = \mu_i^*(T) = \mu_{i_{max}}^*$ and $\frac{\partial J_{PM}}{\partial \mu_i}\Big|_t = \dot{\lambda}_i^*(t) < 0$. To minimize J_{PM} when $t \in [T_s, T]$, μ_i should once again be kept at its maximum value.

Lastly, set $p^* = \phi(\boldsymbol{\mu^*(u^*)})$ to enforce the constraint (6) following Lemma 3. In case $p^* >= 1$, the constraint (6) is infeasible for any $\boldsymbol{\mu(u)}$ with $\boldsymbol{u} \in \mathcal{U}$ and $p \in (0,1)$. Because all feasible perturbations, which only reduce $\boldsymbol{\mu^*}$, can not reduce the value of p even locally (by Lemma 3). In case $p^* <= 0$, this leads to contradiction: $F_{FA}(\boldsymbol{\mu^*}, p^*) <= F_{FA}(\boldsymbol{\mu^*}, 0) = 0$, which contradicts $F_{FA}(\boldsymbol{\mu^*}, p^*) = -\log \alpha > 0$. □

Revision in Appendix Proposition 6 last 4 equations starting from Eqn (24):

$$\begin{aligned} &4(\|x_c - x_t\|^2 - r_t^2) + \\ &+ 8 \Big| \hat{v}^\intercal \ \frac{\epsilon(\|x_c - x_t\|^2 - r_t^2)}{4C_k} \Big(2\frac{x_c - o_i}{\beta_i} + \frac{\alpha_i}{\bar{\beta}_i} \Big) \Big|^2 \\ &= 4\sqrt{J} + \frac{\epsilon^2 J \ |\hat{v}^\intercal \alpha_i|^2}{2C_k^2 \bar{\beta}_i^2} \ . \end{aligned} \quad (24)$$

$$\begin{aligned} &\frac{\hat{v}^\intercal \ \nabla^2 J \ \hat{v}}{16\|x_c - x_t\|^2} \ \nabla \beta_i^\intercal \ \nabla J - 2J \\ &\overset{(24)}{=} \frac{2\sqrt{J} + \frac{\epsilon^2 J}{4C_k^2 \bar{\beta}_i^2} |\hat{v}^T \alpha_i|^2}{8\|x_c - x_t\|^2} \ 2(x_c - o_i) \, 4(x_c - x_t)\sqrt{J} - 2J \\ &= \frac{2J(x_t - o_i)^\intercal (x_c - x_t)}{\|x_c - x_t\|^2} + \frac{\epsilon^2 J^{1.5} |\hat{v}^\intercal \alpha_i|^2 \ (x_c - o_i)^\intercal (x_c - x_t)}{4C_k^2 \bar{\beta}_i^2 \|x_c - x_t\|^2} \end{aligned}$$

$$\begin{aligned} &\frac{\hat{v}^\intercal \ \nabla^2 J \ \hat{v}}{16\|x_c - x_t\|^2} \ \nabla \beta_i^\intercal \ \nabla J - 2J \\ &\le \frac{2J\|x_t - o_i\| \, (\sqrt{\epsilon + \rho_i^2} - \|x_t - o_i\|)}{\|x_c - x_t\|^2} \\ &\quad + \epsilon^2 \sup_{\mathcal{F}_0(\epsilon)} \frac{J^{1.5} |\hat{v}^T \alpha_i|^2 \ (x_c - o_i)^\intercal (x_c - x_t)}{4C_k^2 \bar{\beta}_i^2 \|x_c - x_t\|^2} \ . \end{aligned} \quad (25)$$

$$\begin{aligned} &\frac{\beta^2}{J^{k-1}} \ \hat{v}^\intercal \ \nabla^2 \frac{J^k}{\beta}\Big|_{x_c} \ \hat{v} \\ &\le \frac{2J\bar{\beta}_i \ \|x_t - o_i\| \, (\sqrt{\epsilon + \rho_i^2} - \|x_t - o_i\|)}{\|x_c - x_t\|^2} \\ &+ \epsilon \Bigg(\epsilon \bar{\beta}_i \sup_{\mathcal{F}_0(\epsilon)} \frac{J^{1.5} |\hat{v}^T \alpha_i|^2 \ (x_c - o_i)^\intercal (x_c - x_t)}{4C_k^2 \bar{\beta}_i^2 \|x_c - x_t\|^2} \\ &+ \frac{\hat{v}^\intercal \ \nabla^2 J \ \hat{v}}{16\|x_c - x_t\|^2} \ \nabla \bar{\beta}_i^\intercal \ \nabla J \\ &\qquad + J \ \hat{v}^\intercal \ \Big[\tfrac{1 - \frac{1}{k}}{\bar{\beta}_i} \ \nabla \bar{\beta}_i \, \nabla \bar{\beta}_i^\intercal - \nabla^2 \bar{\beta}_i \Big] \ \hat{v} \Bigg) \ . \end{aligned}$$

Revision in Appendix Proposition 8 last line:

$$\begin{aligned} &\epsilon < \Big(1 - \sqrt{\frac{1 + \zeta^2}{2}} \Big) \frac{\inf_i (\rho_i \inf_{\mathcal{B}_i(\epsilon)} \bar{\beta}_i)}{\sup_{\mathcal{F}_0(\epsilon)} \|\alpha_i\|} \\ &\implies \epsilon < \Big(1 - \sqrt{\frac{1 + \zeta^2}{2}} \Big) \frac{\bar{\beta}_i \|x_c - o_i\|}{\|\alpha_i\|} \end{aligned}$$