On the Controllability of Nearest Neighbor Interconnections

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***Abstract*—In this paper we derive necessary and sufficient conditions for a group of systems interconnected via nearest neighbor rules, to be controllable by one of them acting as a leader. It is indicated that connectivity seems to have an adverse effect on controllability, and it is formally shown why a path is controllable while a complete graph is not. The dependence of the graph controllability property on the size of the graph and its connectivity is investigated in simulation. Results suggest analytical means of selecting the right leader and/or the appropriate topology to be able to control an interconnected system with nearest neighbor interaction rules.**

I. Introduction

Apart from the intellectual challenges of the issues related to control and coordination of large scale interconnected systems, progress has also been fueled by recent technological advances in the areas of embedded computation, wireless communication and microfabrication. Yet, how much can we say that we really know about these systems today? Can we claim that we have satisfactory addressed the fundamental issues in this area? This paper tends to show that there are fundamental issues regarding the interplay between control and communication that are still unresolved and can be formalized into nontrivial new problems.

Significant part of the work associated with control of interconnected systems has been done in the framework of formation control. Lack of space does not let us comment on all of this work, but we can broadly group them in several categories. One class focuses on the effect of interconnection topology on the type of formation equilibrium configurations [1], [2], where it was shown that the rigidity of the interconnection graph plays a crucial role. Another class of coordination algorithms uses potential functions to express the group task [3], [4], [5]. Potential fields have been combined with synchronization control inputs to yield coordinated formation and decentralized flocking and swarming motion [6], [7], [8], [9]. The role of the interconnection topology on information flow and formation stability was investigated in a number of papers [10], [11], [12]. A hybrid system approach was adopted in [13] whereas in [14] graph theory and LMIs are being used. Location optimization problems were examined within a cooperative control framework in [15]. Leader-follower local control laws with vision-based feedback were employed in [16] to stabilize a formation to a particular shape. More reactive, behavioral schemes were employed in [17], [18], [19], [20] to shape formations of vehicles. The investigation of "structural controllability" in [21] is close to the problem discussed in this paper.

In this paper, however, we consider the classical notion of controllability, for a group of autonomous agents interconnected through nearest neighbor rules. We derive conditions on network topology that ensure that the group can be controlled by a particular member which acts as a leader. The rest of the network remains in its original condition, both in terms of interconnection topology and in terms of control laws. It turns out that the answer to the question of whether the group can be controlled depends on the structure of the interconnection topology: the set of all possible interconnection configurations can be partitioned in a class that is controllable and another class that is uncontrollable. We derive algebraic conditions that distinguish and characterize the controllable class and we investigate the dependence of the controllability property on the size of the group and on its connectivity.

The rest of the paper is organized as follows: Section II is a brief review of the graph theoretic terminology used in the paper. Section III follows with an introduction of the interconnected system and concludes with the dynamics of the followers. Our main result is presented in Section IV, in which simulation results are included to verify the analytical derivations. Section V discusses the effect of group size and connectivity on the group controllability and Section VII closes the paper summarizing the discussion and pointing into new research directions.

II. Graph Theory Preliminaries

This section provides a brief introduction to the algebraic graph theoretic objects and their properties that are going to be used in the subsequent analysis. The interested reader is referred to [22] for details.

An (undirected) *graph* $\mathcal{G}$ consists of a *vertex set*, $\mathcal{V}$, and an *edge set* $\mathcal{E}$, where an edge is an unordered pair of distinct vertices in $\mathcal{G}$. If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are said to be *adjacent*, or neighbors and we denote this by writing $x \sim y$. A graph is called *complete* if any two vertices are neighbors. The number of neighbors of each vertex is its *valency* or *degree*. A *path* of length r from vertex x to vertex y is a sequence of $r+1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is said to be *connected*.

The *valency matrix* $\Delta(\mathcal{G})$ of a graph $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, i)-entry is the valency of vertex i. Any undirected graph can be represented by its *adjacency* matrix, $A(\mathcal{G})$, which is a symmetric matrix with $0-1$ elements. The element in position (i, j) (and (j, i) due to symmetry) in $A(\mathcal{G})$ is 1 if vertices i and j are adjacent and 0 otherwise. The symmetric matrix defined as:

$$L(\mathcal{G}) = \Delta(\mathcal{G}) - A(\mathcal{G})$$

is called the *Laplacian* of $\mathcal{G}$. those, is the fact that L is always symmetric and positive semidefinite, and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. For a connected graph, the n-dimensional eigenvector associated with the single zero eigenvalue is the vector of ones, $\mathbf{1}_n$. The second smallest eigenvalue, λ_2 is positive and is known as the algebraic connectivity of the graph, because it is directly related to how the nodes are interconnected. Moreover, it is known that the matrix obtained from $L(\mathcal{G})$ after deleting the row and the corresponding column that is indexed to any vertex, is equal to the number of spanning trees in $(\mathcal{G})$.

III. The Interconnected System and its Leader

Consider N agents with simple, first order dynamics:

$$\dot{x}_i = u_i, \quad i = 1, \ldots, N.$$

The dimension of x could be arbitrary, as long as it is the same for all agents. The analysis that follows is valid for any dimension n, with the difference being that expressions should be rewritten in terms of Kronecker products. For simplicity, we will hereby present the one-dimensional case.

Assume now that each agent is interconnected to a *fixed* number of other agents. Then we can define the *interconnection graph* as follows:

Definition III.1 (Interconnection graph) *The interconnection graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, is being defined as an undirected graph consisting of:*

- *a set of vertices (nodes), $\mathcal{V} = \{n_1, \ldots, n_N\}$, indexed by the agents in the group, and*
- *a set of edges, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V} \mid n_i \sim n_j\}$, containing unordered pairs of nodes that correspond to interconnected agents.*

The indices of the agents that are interconnected to i form its neighboring set, $\mathcal{N}_i = \{j \mid i \sim j\}$. Since this set is fixed, the interconnection graph $\mathcal{G}$ is time invariant. Interconnections are realized through the control inputs, u_i:

$$u_i = -\frac{1}{|\mathcal{N}_i|} \sum_{j \sim i} (x_i - x_j). \tag{1}$$

If we now write the interconnected system in a matrix form, with $x = (x_1, \ldots, x_N)^T$ being the stack vector of all the agent states, we will have:

$$\dot{x} = -\Delta^{-1/2} L \Delta^{-1/2} x, \tag{2}$$

where Δ is the valency matrix of the graph of interconnections and L its Laplacian matrix. Normalizing L in (2) is not crucial, and all the results presented in this paper also hold for the system $\dot{x} = -Lx$.

Remark III.2 (n-dimensional generalizations) *If x was to be considered n-dimensional, and with I_n denoting the n-dimensional identity matrix, then* (2) *would have been written in the form:*

$$\dot{x} = -((\Delta^{-1/2} L \Delta^{-1/2}) \otimes I_n) x.$$

Let us know select an agent arbitrary an agent to be the group leader. Without loss of generality, we can assume that this agent is the one labeled N, and rename the agent states as $z \triangleq x_N$ $y_i \triangleq x_i$, $i = 1, \ldots, N-1$ with y being the stack vector of all y_i. Interconnections with the leader are now assumed unidirectional: the leader's neighbors still obey (1), but the leader is indifferent, and is free to pick u_N arbitrarily. Partitioning (2), we can write the new system in the form:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = -\left[\begin{array}{c|c} F & r \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u_N \end{bmatrix}, \tag{3}$$

where F is the matrix obtained from $\Delta^{-1} L$ after deleting the last row and column, and r is the vector of the first $N-1$ elements of the deleted column. From (3), we extract the dynamics of the followers that correspond to the y component of the equation set:

$$\dot{y} = -Fy - rz. \tag{4}$$

Remark III.3 (Multiple leaders) *One could also consider the case where multiple leaders are selected. In this case, r is going to be a matrix and z will be a vector. Picking more than one leaders offers additional control authority over the group at the expense of the follower population.*

The question that arises now is whether (4) is controllable through z. This will imply that the motion of the leader could bring the group to any desirable configuration. This is the subject of the next section.

IV. Controllable Interconnection Topologies

The controllability matrix of system (4) is

$$C = \begin{bmatrix} -r & Fr & -F^2 r & \cdots & (-1)^n F^{n-1} r \end{bmatrix}$$

Matrix F is symmetric since L is symmetric, and thus it can be expressed as $F = UDU^T$ [23], where the columns of U contain the orthonormal eigenvectors of F, and D is the diagonal matrix of the eigenvalues of F. Then C can be rewritten:

$$C = [-r \quad UDU^T r \quad -(UDU^T)^2 r \quad \cdots \\ \cdots \quad (-1)^n (UDU^T)^{n-1} r]$$

which simplifies to

$$\begin{aligned} C &= [-r \quad UDU^T r \quad -UD^2 U^T r \quad \cdots \\ &\qquad (-1)^n U D^{n-1} U^T r] \\ &= U[-U^T r \quad DU^T r \quad -D^2 U^T r \quad \cdots \\ &\quad \cdots \quad (-1)^n D^{n-1} U^T r] \end{aligned} \tag{5}$$

The rank of the right hand side of (5) is not affected by U because the latter is nonsingular. Therefore, we can focus on the rank of the matrix multiplying from the right:

$$[-U^T r \quad DU^T r \quad -D^2 U^T r \quad \cdots \\ \cdots \quad (-1)^n D^{n-1} U^T r], \tag{6}$$

which is the controllability matrix of the (decoupled) system $\dot{q} = -Dq - U^T r z$. Since D is a diagonal nonsingular matrix (due to the matrix-tree Theorem), the effect of D multiplying a vector will be a scaling along each of its dimensions. From this form it is clear that a necessary condition for (6) to be full rank is that *all elements of $U^T r$ are nonzero;* if one is zero then a whole row in (6) will be zero and the matrix will be rank-deficient. In addition, for the system to be controllable, *no two elements of D can be the same;* otherwise one could be trying to steer the same dynamics, with the same (scaled) input to different states - in (6) this would appear as one row being the scaled version of another.

We summarize this discussion in the following Theorem:

Theorem IV.1 (Controllable Topologies) *Consider an interconnected system described by (2) that corresponds to a connected interconnection graph with Laplacian L. Let $z = x_k$ be one of the states of (2) and let F be the matrix obtained by $\Delta^{-1} L$ after deleting the k^{th} row and column. Let r be the the vector derived from the k^{th} column of $\Delta^{-1} L$ after deleting the k^{th} element. Then z can control the dynamics of all other states in x if and only if the following conditions are satisfied:*

1. *The eigenvalues of F are all distinct;*
2. *The eigenvectors of F are not orthogonal to r.*

From Theorem IV.1 it is clear that the topology of the interconnection graph completely determines its controllability properties. The conditions of the Theorem partition the set of all connected interconnection graphs into a controllable and an uncontrollable class. The lack of a graph theoretic characterization of these classes at this point prevents us from being able to *construct* controllable interconnection topologies. Developing such a characterization is along the directions of our current research.

A. Numerical Verification of Theorem IV.1

In this section we present simulation results that show how the interconnected systems can be steered to specific positions by regulating the motion of a single system that plays the role of the group leader. The interconnection graph consists of ten nodes, (Figure 1) and node 10 is being used as a leader. The objective is move the leader so that the remaining 9 nodes are in successively steered into configurations which outline the three letters: U, N, M.

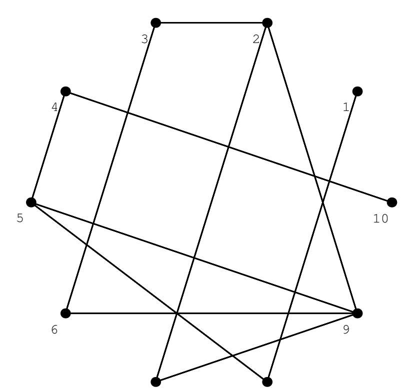

Fig. 1. *The controllable interconnection graph used in simulations. Node 10 is the leader.*

The interconnection graph that corresponds to the system at hand has the following Laplacian:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 3 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The spectrum of F, which for simplicity it is taken without the Δ-scaling are:

$$\sigma(F) = \{5.47633, 4.08251, 3.58315, 2.50829, 2.35533, 1.44384, 1.22489, 0.278455, 0.0471964\}$$

It is immediately seen that they are distinct. As for the eigenvectors of F, we have

$$U^T r = (-0.1166, -0.2315, -0.2804, -0.1154, 0.5721, 0.3211, -0.6182, 0.0963, -0.1376)$$

giving a vector that is element-wise nonzero. Thus the requirements of Theorem IV.1 are satisfied.

The nodes start from random initial positions as depicted in Figure 2. They are shown as filled black dots

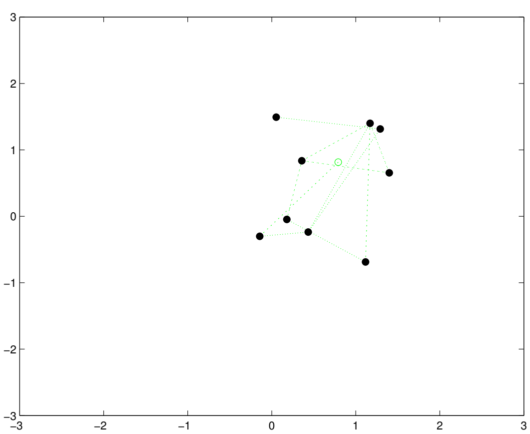

Fig. 2. *Initial node configurations.*

with the exception of the leader which is depicted as a (green) circle. Interconnections are marked as dotted fainted (green) lines connecting the corresponding nodes. Starting from this initial configuration, the nodes are initially steered to the configuration of Figure 3. Figure 4 shows the evolution of the position errors, from the initial (random) configuration to that which forms the letter U. This is the initial configuration for the second run, which drives the nodes to form the next letter, N, at steady state (Figure 5). The time history of these position errors is given in Figure 6. Similarly, N is the initial state for the third run which steers the nodes to form the last letter, M, shown in Figure 7, along trajectories the errors of which are given in Figure 8.

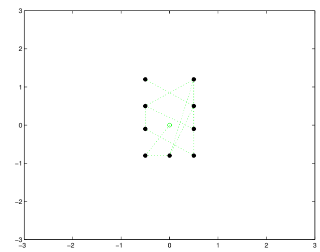

Fig. 3. *The letter U.*

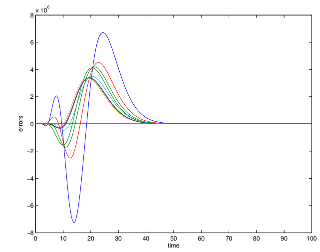

Fig. 4. *Position errors.*

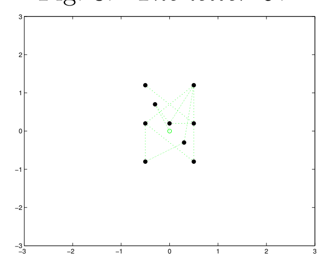

Fig. 5. *The letter N.*

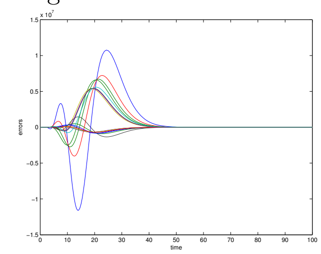

Fig. 6. *Position errors.*

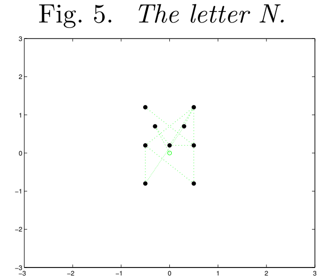

Fig. 7. *The letter M.*

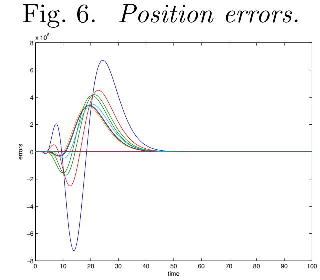

Fig. 8. *Position errors.*

V. Connectivity is Not Always Good

It is surprising that increased connectivity has an adverse effect on the controllability of the network. At the two far ends of the connectivity scale we have systems that are either always controllable or always uncontrollable.

Proposition V.1 *A complete graph K_N is uncontrollable.*

Proof: The Laplacian of a complete graph is given as $L = N \cdot I_N - \mathbf{1}_N^T \mathbf{1}_N$, with I_N denoting the N-dimensional identity matrix and $\mathbf{1}_N$ the N-dimensional vector of ones. No matter what column is deleted, F will still be expressed as $I_{N-1} - \frac{1}{N}\mathbf{1}_{N-1}^T \mathbf{1}_{N-1}$. Thus, the spectrum of F will be $\{\frac{1}{N}, 1^{(N-2)}\}$, so unless $N = 2$ (trivial case), eigenvalue 1 will have multiplicity and the graph will be uncontrollable according to Theorem IV.1. ■

At the other end of the connectivity range, things look more optimistic:

Proposition V.2 *A path P_N is controllable.*

Proof: By renaming the vertices we can always write the Laplacian of P_N into the following form:

$$L_{P_N} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

This matrix is easily identified as the opposite of a Jacobi matrix. For Jacobi matrices the eigenvalues are real and distinct. Matrix F is constructed from the $N-1$ leading principal submatrix of L_{P_N}, $\bar{L}_{P_N}$. The eigenvalues of a the leading principal submatrix of Jacobi matrix separate those of the Jacobi matrix, which means that they are distinct two. Thus, $\Delta^{-1/2}\bar{L}_{P_N}\Delta^{-1/2}$ has distinct eigenvalues. Notice that:

$$\bar{L}_{P_N}x = (x_1 - x_2, -x_1 + 2x_2 + x_3, -x_2 + 2x_3 + x3, \ldots \\ \ldots, -x_{N-2} + 2x_{N-1} + x_N, -x_{N-1} + 2x_N).$$

If x is an eigenvector affording an eigenvalue λ, then

$$\lambda(x_1, \ldots, x_N) = (x_1 - x_2, -x_1 + 2x_2 + x_3, \ldots \\ \ldots, -x_{N-2} + 2x_{N-1} + x_N, -x_{N-1} + 2x_N).$$

If $x_N = 0$, then it is easily seen that the equality propagates and results to:

$$x_1 = x_2 = \cdots = x_N = 0$$

which is a contradiction. Thus, in all eigenvectors of $\bar{L}_{N-1}$, the last element is nonzero. In this case, given that $r = (0, 0, \ldots, 0, -1)$, it is clear that $r \cdot x \neq 0$ for all eigenvectors x of $\bar{L}_{N-1}$. Therefore, both conditions of Theorem IV.1 are satisfied and P_N is controllable from either end-node. ■

VI. Size Does Matter

Lacking a graph theoretic characterization of the class of controllable graphs, that would enable us to *construct* controllable graphs instead of checking their controllability a posteriori, we run simulations to assess the effect of the size of the graph on its controllability properties.

In these simulations, we constructed random graphs of different sizes which we then tested for controllability. We distinguished between graphs with different edge probability, since we determined in the previous section that connectivity, and therefore edge density, has an adverse effect on controllability. For a given edge probability we constructed a large number of graphs and we recorded the number of iterations needed each time before a controllable graph was discovered. Then we took the mean over twenty iteration samples. Figure 9 depicts the general trend of these mean values with respect to the graph size and the edge probability.

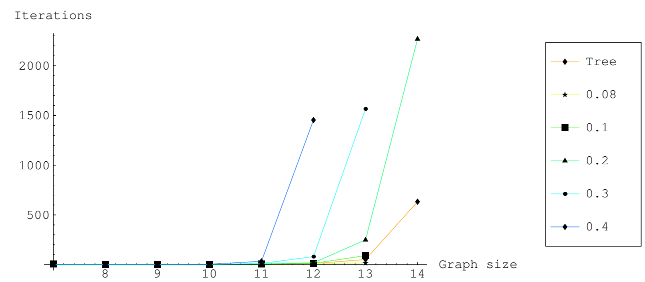

Fig. 9. *Probability of finding a controllable random graph for different probabilities for edge occurrence and graph sizes. The numbers on the legend on the right give the probabilities for edge occurrence in each random graph. The first is the case of a tree (minimum connectivity)*

Simulations verify that it is unlikely for a graph with increased connectivity to be controllable. This can be seen in Figure 9 from the fact that as the edge occurrence probability increases, the corresponding iteration curve moves upwards. It is notable that for edge probability greater or equal to 0.3, it was not possible to find a controllable graph for $N = 13$ in a reasonable number of trials. The simulations also indicate that the probability of finding a controllable graph within the set of random graphs with certain probability decreases exponentially with the size of the graph. For graphs of size larger than $N = 14$ it proved extremely difficult to obtain controllable samples, regardless of connectivity.

These results emphasize the importance of a graph theoretic characterization of the class of controllable interconnections, which will enable the justification of the effect of graph size through a formal analysis.

VII. Conclusions

In this paper we demonstrated how the structure of the interconnection topology in a group of systems linked with nearest neighbor interaction rules can affect the controllability of the overall interconnected system. We derived necessary and sufficient conditions that enable the states of interconnected subsystems to be controlled from a single one acting as a leader. These conditions are translated to algebraic tests on the eigenvalues and eigenvectors of the submatrix of the graph's Laplacian, which corresponds to the followers in the group. It was also demonstrated, both in theory and in simulation that increased graph connectivity does not necessarily improve the controllability of the interconnected system. The lack of a graph theoretic characterization of the controllability property prevents us from building controllable interconnection topologies and the probability of discovering such topologies seems to decrease with the size of the graph. Further research is directed on constructive characterizations of controllability in such interconnected systems, where interaction is based on local, nearest neighbor rules.

References

[1] P. N. B. Tolga Eren and A. S. Morse, "Closing ranks in vehicle formations based on rigidity," in *Proceedings of the IEEE Conference on Decision and Control*, Las Vegas, NV, December 2002, pp. 2959–2964.

[2] R. Olfati-Saber and R. M. Murray, "Graph rigidity and distributed formation stabilization of multi-vehicle systems," in *Proceedings of the IEEE International Conference on Decision and Control*, Las Vegas, NV, December 10-13 2002, pp. 2965–2971.

[3] P. Ögren, E. Fiorelli, and N. E. Leonard, "Formations with a mission: Stable coordination of vehicle group maneuvers," in *Proceedings of the 15th International Symposium on Mathematical Theory of Networks and Systems*, Notre Dame, IN, August 2002.

[4] P. Ögren, M. Egerstedt, and X. Hu, "A control Lyapunov function approach to multiagent coordination," *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, pp. 847–851, October 2002.

[5] V. Gazi and K. M. Passino, "Stability analysis of swarms," *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 692–696, April 2003.

[6] E. Justh and P. Krishnaprasad, "A simple control law for UAV formation flying," Institute for Systems Research, Tech. Rep. 2002-38, 2002.

[7] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stable flocking of mobile agents, Part I: Fixed topology," in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2010–2015.

[8] ——, "Stable flocking of mobile agents, Part II: Dynamic topology," in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2016–2021.

[9] R. O. Saber and R. M. Murray, "Flocking with obstacle avoidance: Cooperation with limited communication in mobile networks," in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2022–2028.

[10] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," in *Proceedings of the 15th IFAC World Congress*, Barcelona, Spain, July 2002, pp. 2360–2365.

[11] H. G. Tanner, G. J. Pappas, and V. Kumar, "Leader-to-formation stability," *IEEE Transactions on Robotics and Automation*, 2002, (to appear).

[12] L. B. Cremean and R. M. Murray, "Stability analysis of interconnected nonlinear systems under matrix feedback," in *Proceedings of 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, December 2003, pp. 3078–3083.

[13] S. Zelinski, J. Koo, and S. Sastry, "Hybrid system design for formations of autonomous vehicles," in *Proceedings of 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, December 2003, pp. 1–6.

[14] M. Mesbahi and F. Hadaegh, "Formation flying of multiple spacecraft via graphs, matrix inequalities, and switching," *AIAA Journal of Guidance, Control and Dynamics*, vol. 24, no. 2, pp. 369–377, March 2001.

[15] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, 2002, (submitted).

[16] A. K. Das, R. Fierro, V. Kumar, J. P. Ostrowski, J. Spletzer, and C. J. Taylor, "A vision-based formation control framework," *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, pp. 813–825, October 2002.

[17] J. Fredslund and M. J. Matarić, "A general algorithm for robot formations using local sensing and minimal communications," *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, pp. 837–846, October 2002.

[18] T. Balch and R. C. Arkin, "Behavior-based formation control for multirobot teams," *IEEE Transactions on Robotics and Automation*, vol. 14, no. 6, pp. 926–939, December 1998.

[19] E. Klavins, "Automatic synthesis of controllers for distributed assembly and formation forming," in *Proceedings of IEEE International Conference on Robotics Automation*, Washington, DC, May 2002, pp. 3296–3302.

[20] J. R. T. Lawton, R. W. Beard, and B. J. Young, "A decentralized approach to formation maneuvers," *IEEE Transactions on Robotics and Automation*, vol. 19, no. 6, pp. 933–941, December 2003.

[21] J. T. Feddema, C. Lewis, and D. A. Schoenwald, "Decentralized control of cooperative robotic vehicles: theory and application," *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, pp. 852–864, October 2002.

[22] C. Godsil and G. Royle, *Algebraic Graph Theory.* New York: Springer Graduate Texts in Mathematics # 207, 2001.

[23] R. Horn and C. R. Johnson, *Matrix Analysis.* New York: Cambridge University Press, 1985.