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Relaxed Stability Conditions for Switched Systems with Dwell Time

Herbert G. Tanner

ABSTRACT

The paper presents asymptotic, and input-to-state stability results, for switched systems with dwell time in which the switching signal is not arbitrary, but is rather chosen as part of the control design strategy. Then, appropriate switching policies allow the use of functions without sign-definite time derivatives in lieu of a common Lyapunov-like function.

I. INTRODUCTION

Stability certificates for nonlinear switched systems often come in the form of Lyapunov functions, common or multiple. A review on stability results under arbitrary, and constrained switching, are available in the context of switched [1], and hybrid systems [2]. For randomly switching nonlinear systems, stability can be analyzed using multiple Lyapunov functions [3]. Conditions for establishing uniform stability in nonlinear time-varying switched systems are provided in [4]. For uniform asymptotic stability within the Lyapunov framework, an upper bound on the derivative of the Lyapunov function in the form of a negative definite function is usually required [1, 2, 5].

In the case of a common Lyapunov function, in particular, it is known [1, Part II] that the supremum of the Lyapunov function derivative cannot be taken as zero when switching is arbitrary. One might be tempted to apply an invariance argument, and in the context of switched nonlinear systems related results exist for systems with state-dependent dwell time [6], average dwell time [7, 8], and persistent or weak dwell-time [9]. There are some limitations, though. In this work [6, 8, 7] the index set for the switched system is finite, and typically to invariance-type results, determining the invariant set may be challenging. Specifically, to verify weak invariance as in [6], one may have to check all solutions for every element of the index set in the worst case. This is clearly infeasible if the index set is not finite.

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Mechanical Engineering, University of Delaware, Newark, DE 19808, USA
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For the case where the time derivative of the Lyapunov-like functions, common or multiple, is sign-indefinite, there exist results for switched linear systems [10, 11]. Some related work can also be found in the context of hybrid systems for single [12] and multiple [2] Lyapunov functions, and single Lyapunov functions for discontinuous systems [13]. The idea behind multiple Lyapunov functions [2] is that when a system that flows along a given vector field switches to another vector field, the Lyapunov function associated with the original vector field has to decrease when that same vector field is reactivated; in the meantime, other vector fields somehow overcome the increase of that particular function, in addition to decreasing their own Lyapunov function. This reasoning does *not* directly apply to the single function case, since it would imply that the function decreases *all* the time. The idea behind a single function [12], on the other hand, is that the function decreases at *switching times*, and in the meantime is bounded by a continuous [12] or linear [13] function of its value at the last switching instant.

In this paper, a result similar in spirit to one established in [12] is first presented. Compared to that of [12, Theorem 4.1], the one in this paper is less general, because the rate of change of the function is bounded by a $\mathcal{K}$-class instead of just a continuous function; the more stringent conditions, however, allow us an input-to-state stability (ISS) extension. Compared to the stability result of [13], the one presented here is less conservative, because linear comparison functions are only a subclass of the $\mathcal{K}$-class functions of this paper. In addition, what is implicitly assumed by [12, 13], is emphasized in this paper: instead of a common Lyapunov function, the switching signal can be used to establish the stability properties of the switched system.

The idea of using switching to control an otherwise unstable or uncontrollable system has already found application in the context of switched linear [14] as well as event-triggered systems, both linear [15] and nonliner [16]. Another case where switching can be controlled, and a function with a sign-indefinite time derivative may be needed, can be found in applications of Model Predictive Control (MPC) to mobile robot navigation [17, 18, 19], as well as in the context of self-triggered control [15]. In both of these cases, there is also a need to establish robustness with respect to external disturbances.

Some form of robustness can be established in the context of *input to state stability* (ISS). For nonlinear switched systems, ISS is typically established by ensuring that each component dynamics is ISS [20, 21]. (Different assumptions are made for feedforward systems by [22].) The result in this paper is different, in the sense that the individual component dynamics need not be ISS; the ISS conditions we impose on them apply only during the time interval when the particular component dynamics is active. Along similar lines, results for composite input/output-to-state systems are proposed by [23]. Another feature that differentiates the approach described here from related literature is the fact that the switching signal is used as a control design tool for nonlinear systems. Trading-off the generality of switching allows us to relax the sign-definiteness of the Lyapunov-like function derivative, without affecting asymptotic stability. In this context, the switching signal becomes a mechanism for stabilization, and functions that do not qualify as Lyapunov functions can be used as stability certificates.

In summary, the contribution of this paper is twofold: a) it relaxes the conditions on the derivative of a Lyapunov-like function of a switched system by controlling the switching signal, and b) it does so in a way that input-to-state stability can be established without relying on the component dynamics being ISS. The results suggest that given a monotonically increasing sequence of switching instants,* separated by at least some constant dwell time, stability is recovered if some Lyapunov-like function evolves fairly liberally *between*, but decreases sufficiently *at* switching times.

Section II presents the asymptotic stability result for switched systems with a common Lyapunov-like function. The same section contains a regional ISS [24] extension of this result. Section III offers a detailed computational example of the application of the proposed framework on physical systems. Finally, Section IV summarizes the results of the paper.

II. MAIN RESULTS

Following [1], we consider the switching system

$$\dot{x} = f_{\sigma(t)}(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $\sigma : \mathbb{R}_+ \to \mathbb{N}_+;\ t \mapsto p$ is a piecewise constant signal that indicates the vector field from a parameterized family $\{f_p : p \in \mathbb{N}_+\}$† which is active at a given time instant $t \in \mathbb{R}_+$. We assume that each $f_{\sigma(t)}$ satisfies the standard Lipschitz continuity conditions for all $p \in \mathbb{N}$, and the solutions of (1) are defined in the Caratheodory sense as in [1].

The switching signal belongs in a class denoted $\mathcal{S}^+_{\mathsf{dwell}}$ (cf. [5]) defined as follows.

Definition 1 (switching signals) *The set* $\mathcal{S}^+_{\mathsf{dwell}}$ *contains all piecewise constant functions from* $\mathbb{R}_+$ *onto* $P \subset \mathbb{N}_+$*, for which the inverse image of every* $p \in P$ *is a (possibly infinite) collection of intervals in* $\mathbb{R}_+$ *of measure no less than* $\tau_D > 0$ *and no more than* $T_D < \infty$.

Switching signals in $\mathcal{S}^+_{\mathsf{dwell}}$ induce ascending sequences $\{\tau_i\}_{i\in\mathbb{N}}$ of switching times (which include instances at which $\sigma(t)$ is discontinuous [5]), with the property that $\forall i \in \mathbb{N}$, $\tau_{i+1} - \tau_i \geq \tau_D$. Note also that the domain of σ is the whole $\mathbb{R}^+$ implying that for

*Actual switching may not necessarily occur at all these times (see Section II).

†Since p can be arbitrarily large, this collection can potentially have infinite many members.

all $t \in \mathbb{R}_+$, there exists a $p \in \mathbb{N}$ such that $p = \sigma(t)$. The case of systems in which switching stops can be accounted for by considering $\sigma \in \mathcal{S}^+_{\text{dwell}}$ which are ultimately periodic, i.e., they exhibit periodicity after $k < \infty$ switches and for which $f_{\sigma(\tau_i)} = f_{\sigma(\tau_j)}$, $\forall\, i, j > k$.

Theorem 1 *Let $V(x)$ be a differentiable, positive definite function in a domain $\mathcal{D} \subseteq \mathbb{R}^n$ that contains the origin. For some class-$\mathcal{K}$ functions [25] α and γ, and within each of the switching intervals $(\tau_i, \tau_{i+1}]$ of some $\sigma \in \mathcal{S}^+_{\text{dwell}}$, assume that every solution of (1) satisfies*

$$\sup_{t\in(\tau_i,\tau_{i+1}]} \left\{ \tfrac{\partial V}{\partial x} f_{\sigma(t)}(x) \right\} < \frac{\alpha(V\left(x(\tau_i)\right))}{\tau_{i+1} - \tau_i} \qquad (2)$$

$$V\big(x(\tau_{i+1})\big) - V\big(x(\tau_i)\big) \le -\gamma(\|x(\tau_i)\|) \ . \qquad (3)$$

Then all trajectories of (1) starting in $\mathcal{D}$ asymptotically converge to the origin. Moreover, if $\mathcal{D} = \mathbb{R}^n$, the system is globally asymptotically stable.

Proof : For the arbitrary time interval (τ_i, τ_{i+1}), pick an $\varepsilon > 0$ and define a ball $B_\varepsilon \triangleq \{x \in \mathcal{D} \mid \|x\| \le \varepsilon\}$ (Fig. 1). On the boundary of B_ε, $V(x)$ attains a minimum by continuity. Call this minimum $V_\varepsilon \triangleq \min_{x\in\partial B_\varepsilon} V(x)$ and define $\Omega_\varepsilon \triangleq \{x \in \mathcal{D} \mid V(x) \le V_\varepsilon\}$.

Select an appropriately small $\delta > 0$ so that $r \triangleq V_\varepsilon - \alpha(\delta) > 0$, and define $\Omega_\delta \triangleq \{x \in \mathbb{R}^n \mid V(x) \le \min\{r, \delta\}\}$. Then inside Ω_δ, fit a ball $B_\delta \triangleq \{x \in \mathbb{R}^n \mid \|x\| \le \min_{x\in\partial\Omega_\delta} \|x\|\}$.

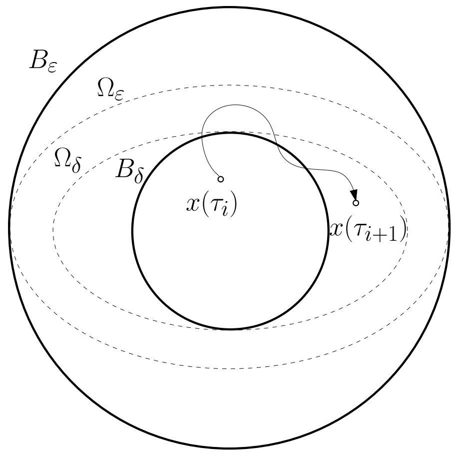

Fig. 1. The sets defined in the proof of Theorem 1, and a possible system trajectory in the time interval $[\tau_i, \tau_{i+1}]$.

Let $x(0) = \tau_1$ be in B_δ. Then for $t \in (\tau_1, \tau_2)$ $V(x(t)) = V(x(\tau_1)) + \int_{\tau_1}^{t} \frac{\partial V}{\partial x} f_{\sigma(t)}(x(\tau))\, \mathrm{d}\tau$, and due to (2)

$$\begin{aligned} V(x(t)) &\le V(x(\tau_1)) + \frac{\alpha\big(V(x(\tau_1))\big)}{\tau_2 - \tau_1}(t - \tau_1) \\ &\le V(x(\tau_1)) + \alpha\big(V(x(\tau_1))\big) \\ &\le V(x(\tau_1)) + \alpha\big(\min\{r, \delta\}\big) \le r + \alpha(\delta) < V_\varepsilon, \end{aligned}$$

which means that for $t \in (\tau_1, \tau_2]$, $x(t)$ remains in Ω_ε. Now we have $x(\tau_2) \in \Omega_\varepsilon$, and since (3) implies $V(x(\tau_3)) < V(x(\tau_2))$, we have

$$\begin{aligned} V(x(t)) &\le V(x(\tau_2)) + \alpha\big(V(x(\tau_2))\big) \\ &\le V(x(\tau_1)) + \alpha\big(V(x(\tau_1))\big) < V_\varepsilon \ . \end{aligned}$$

Repeating the exact same argument for $x(\tau_i) \in B_\delta$ with $i > 2$, we conclude that for $t \in (\tau_i, \tau_{i+1})$, $x(t) \in \Omega_\varepsilon$. This establishes the positive invariance of Ω_ε in an inductive way.

Since we know that at every τ_i, (1) satisfies (2) and (3), a similar inductive argument can be applied to establish the stability of the system at the origin.

Condition (3) indicates a converging sequence $\{V\big(x(\tau_k)\big)\}$, since $V(x)$ is lower bounded by zero, which is strictly decreasing. Since the sequence is converging, it is a Cauchy sequence and therefore $\|V\big(x(\tau_i)\big) - V\big(x(\tau_{i+1})\big)\| \to 0$ when $i \to \infty$. Rewriting (3) as $V\big(x(\tau_{i+1})\big) - V\big(x(\tau_i)\big) \geq \gamma\big(\|x(\tau_i)\|\big)$, we see that the convergence of $\|V\big(x(\tau_i)\big) - V\big(x(\tau_{i+1})\big)\|$ to zero forces $\gamma\big(\|x(\tau_i)\|\big) \to 0$ as $i \to \infty$. Given that γ is a class $\mathcal{K}$ function, that can only happen if and only if $\|x(\tau_i)\| \to 0$.[‡] ∎

Theorem 1 implies that if a dwell-time switching signal can be *selected* to make an otherwise failing Lyapunov function candidate decrease sufficiently (measured by some $\mathcal{K}$-class function γ) at switching times, then asymptotic stability can be recovered. Conditions (2)–(3) are not meant to be verified analytically. Since $\frac{\partial V}{\partial x} f_p(x)$ is a scalar, straightforward constrained numerical optimization can offer bounds for this Lie derivative within compact subsets of the state space. While no general method is offered here, depending on the shape of the boundary of state space regions where the derivative has the "wrong" sign, a combination of numerical Lie derivative extremum computation with applications of the comparison lemma may indicate switching signals for which (2)-(3) is satisfied (see Section III).

[‡]Note how the role of the class-$\mathcal{K}$ functions that uniformly bound V from above and below in classical statements of Lyapunov's second method, is played here by γ. The system therefore is asymptotically stable at the origin.

Assume now that (1) is augmented with affine "inputs," which can be thought of as disturbances or unmodeled dynamics:

$$\dot{x} = f_{\sigma(t)}(x) + u(t). \tag{4}$$

The assumption on $u(\cdot) : [0, \infty) \to \mathbb{R}^n$ is that it is a measurable and essentially bounded function. Let $\|\cdot\|_\infty \triangleq \sup_{t>0}\{\cdot\}$ denote the essential supremum of a function over time.

Definition 2 (cf. [24]) *Given a compact set $\mathcal{D}$, the interior of which contains the origin, (4) is said to be (regionally) input-to-state stable (ISS) in $\mathcal{D}$, if some $\Xi \supseteq \mathcal{D}$ is positively invariant for (4) with $u(t) \in \mathcal{U} \subseteq \mathbb{R}^n$, and there exist a $\mathcal{KL}$-class function β and a $\mathcal{K}$-class function c such that*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + c(\|u\|_\infty), \tag{5}$$

for $x(0) \in \mathcal{D}$ and $u \in \mathcal{U}$.

Now we can state the following robustness result.

Theorem 2 *Let $V(x)$ be a differentiable, positive definite function satisfying*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

in a domain $\mathcal{D} \subseteq \mathbb{R}^n$ that contains the origin, for some class-$\mathcal{K}$ functions α_1 and α_2, with α_1 being Lipschitz. If there exist class-$\mathcal{K}$ functions α, γ, and

$\zeta(\cdot)$, and a switching signal $\sigma \in \mathcal{S}^{+}_{\text{dwell}}$ such that on each of the switching intervals $(\tau_i, \tau_{i+1}]$ of σ, every admissible solution in $\mathcal{D}$ satisfies

$$\sup\left\{\frac{\partial V}{\partial x}(f_{\sigma(t)}(x)+u)\right\} < \frac{\alpha\big(V(x(\tau_i)) + \|u\|_\infty\big)}{\tau_{i+1}-\tau_i} \quad (6)$$

$$V\big(x(\tau_{i+1})\big) - V\big(x(\tau_i)\big) \leq -\gamma\big(V(x(\tau_i))\big) + \zeta(\|u\|_\infty) \ , \quad (7)$$

then (4) is regionally ISS *in $\mathcal{D}$.*

Proof : Let $\Omega_0 \triangleq \{x \in \mathcal{D} \mid \zeta(\|u\|_\infty) + \varepsilon \geq \gamma(V(x))\}$, where $\varepsilon > 0$ a (small) constant. Denote V_0 the value of $V(x)$ for which $\gamma(V(x)) = \zeta(\|u\|_\infty) + \varepsilon$.

For $x(\tau_i) \notin \Omega_0$ we can write

$$V(x(\tau_{i+1})) \leq V(x(\tau_i)) - \gamma(V(x(\tau_i))) + \zeta(\|u\|_\infty) < V(x(\tau_i)) - \varepsilon \ . \quad (8)$$

Note that for $x(\tau_i) \notin \Omega_0$ such that $V\big(x(\tau_i)\big)$ is sufficiently large, the right-hand side of (7) becomes negative, ensuring the boundedness of $V\big(x(t)\big)$. It follows from the positive definiteness and radial unboundedness of V that the dynamics is forward complete. Knowing that between consecutive switching times V decreases by at least ε, allows us to define intervals of variation for $\|x(\tau_i)\|$:

$$\|x(\tau_i)\| \in \left[\alpha_2^{-1}\big(V(x(\tau_i))\big), \alpha_1^{-1}\big(V(x(\tau_i))\big)\right], \quad (9)$$

as shown in Fig. 2. The maximum values in the intervals defined for consecutive switching times τ_i and τ_{i+1} are separated by at least $\frac{\varepsilon}{L}$, where L is the Lipschitz constant of α_1. Note that although $\|x(t)\|$ might increase between switchings, the interval bounds have to decrease (Fig. 2) because they depend on the value of V. We can actually find the number k of switchings after which one can ensure that $\|x(\tau_{i+k})\|$ will be smaller than $\|x(\tau_i)\|$:

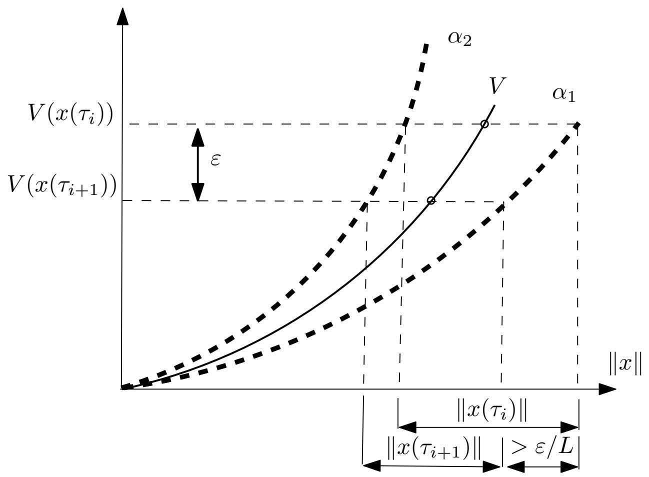

Fig. 2. The value of V at switching times constrains the norm of the state at these instants in terms of the inverses of the bounding functions. The intervals indicated on the horizontal axis denote the range of possible values for $\|x\|$ if a common level set for α_1^{-1} and α_2^{-1} is given.

$$V(x(\tau_{i+1})) \leq V(x(\tau_i)) - \gamma(V(x(\tau_i))) + \zeta(\|u\|_\infty) < V(x(\tau_i)) - \varepsilon$$

$$V(x(\tau_{i+2})) \leq V(x(\tau_{i+1})) - \gamma(V(x(\tau_{i+1}))) + \zeta(\|u\|_\infty) < V(x(\tau_{i+1})) - \varepsilon$$

$$\vdots$$

$$V(x(\tau_{i+k})) \leq V(x(\tau_{i+k-1})) - \gamma(V(x(\tau_{i+k-1}))) + \zeta(\|u\|_\infty) < V(x(\tau_{i+k-1})) - \varepsilon$$

and summing the inequalities, get $V(x(\tau_{i+k})) < V(x(\tau_i)) - k\varepsilon$. There is a finite $k = k_i$ for which $V(x(\tau_i)) - k\varepsilon < \alpha_1(\|x(\tau_i)\|)$, from which it follows

$$\alpha_1(\|x(\tau_{i+k_i})\|) \le V(x(\tau_{i+k_i})) < V(x(\tau_i)) - k_i\varepsilon < \alpha_1(\|x(\tau_i)\|) \ . \quad (10)$$

From the right hand side of (10) we get $k_i > \frac{V(x(\tau_i)) - \alpha_1(\|x(\tau_i)\|)}{\varepsilon}$, and after noting that the difference between $V(x)$ and $\alpha_1(\|x\|)$ is always upper bounded by $\alpha_2(\|x\|)$ we can select $k_i = \left\lceil \frac{\alpha_2(\|x(\tau_i)\|)}{\varepsilon} \right\rceil$. Note now that as $\|x(\tau_i)\|$ decreases, k_i decreases too, since α_2 is a class-$\mathcal{K}$ function. Therefore, for all $i > 0$ the integer

$$k \triangleq k_0 = \left\lceil \frac{\alpha_2(\|x(0)\|)}{\varepsilon} \right\rceil$$

corresponds to a number of switching times sufficient to bring the state closer to the origin for any switching instant $\tau_i \ge 0$.

We can thus guarantee that states at switching instants with indices that are k apart satisfy $\|x(\tau_i)\| - \|x(\tau_{i+k})\| > \frac{k\,\varepsilon}{L}$, where L is the Lipschitz constant of α_1. Within k switching intervals, say from τ_i to τ_{i+k}, $\|x(t)\|$ may increase by at most

$$\rho_i = \alpha_1^{-1}\left(\tfrac{2T_D}{\tau_D}\alpha\big(2\alpha_2(\|x(\tau_i)\|)\big) + 2\alpha_2\big(\|x(\tau_i)\|\big)\right) + \alpha_1^{-1}\left(\tfrac{2T_D}{\tau_D}\alpha(2\|u\|_\infty)\right) \quad (11)$$

compared to $\|x(\tau_i)\|$, but will eventually be reduced by at least $\frac{\varepsilon k}{L}$ with respect to $\|x(\tau_i)\|$ (Fig. 3).

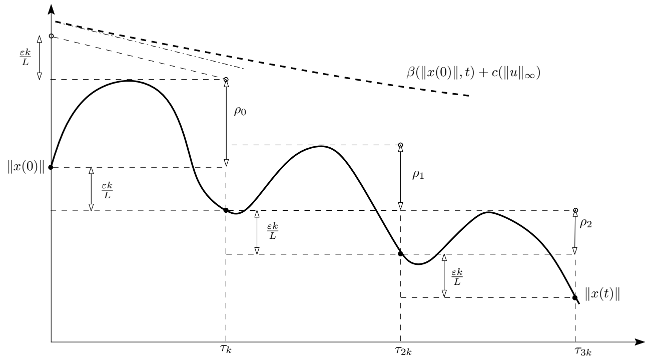

Fig. 3. The construction of the comparison functions for the proof of Theorem 2.

Given that $\varepsilon + \zeta(\|u\|_\infty) = \gamma(V_0)$, it follows that $\varepsilon \le \gamma\big(\alpha_2(\|x(0)\|)\big) + \zeta(\|u\|_\infty)$, and therefore

$$\|x(0)\| + \tfrac{\varepsilon k}{L} + \rho_0 \le \|x(0)\| + \tfrac{k}{L}\Big(\gamma\big(\alpha_2(\|x(0)\|)\big) + \zeta(\|u\|_\infty)\Big) + \rho_0 \ .$$

Combining with (11) and after some manipulation,

$$\|x(0)\| + \tfrac{\varepsilon k}{L} + \rho_0 \le \|x(0)\| + \tfrac{k}{L}\gamma\big(\alpha_2(\|x(0)\|)\big) + \alpha_1^{-1}\left(\tfrac{2T_D}{\tau_D}\alpha\big(2\alpha_2(\|x(0)\|)\big) + 2\alpha_2(\|x(0)\|)\right) + \tfrac{k}{L}\zeta(\|u\|_\infty) + \alpha_1^{-1}\big(\tfrac{2T_D}{\tau_D}\alpha(2\|u\|_\infty)\big) \ .$$

With ρ_i being a class $\mathcal{K}$ function of $\|x(\tau_i)\|$, and with $\|x(\tau_{i+k})\|$ unable to increase, it follows that $\{\rho_i\}_{i=0}^n$ is a monotonically decreasing sequence. Therefore, the sequence $\|x(\tau_{nk})\| + \frac{k\varepsilon}{L} + \rho_n$ is decreasing at an accelerating rate, and that rate never falls below $\frac{\varepsilon}{LT_D}$

(Fig. 3). It follows that the function

$$\Big[\|x(0)\| + \tfrac{k}{L}\gamma\big(\alpha_2(\|x(0)\|)\big) + \alpha_1^{-1}\left(\tfrac{2T_D}{\tau_D}\alpha\big(2\alpha_2(\|x(0)\|)\big) + 2\alpha_2(\|x(0)\|)\right)\Big]\mathrm{e}^{\frac{\varepsilon t}{LT_D}} + \tfrac{k}{L}\zeta(\|u\|_\infty) + \alpha_1^{-1}\big(\tfrac{2T_D}{\tau_D}\alpha(2\|u\|_\infty)\big)$$

always bounds the evolution of $\|x(t)\|$ from above. The first term in the above is a class $\mathcal{KL}$ function of the form $\beta(\|x(0)\|, t)$, while the two next form a class $\mathcal{K}$ function of $\|u\|_\infty$.

∎

III. Case study

This section aims at demonstrating the application of Theorems 1 and 2 to the stability analysis of a physical systems that have been studied in recent literature [26]. Specifically, the discussion illustrates how numerical calculations—not explicit solution of differential equations—and the use of the comparison principle, allows the construction of the bounding functions appearing on the right-hand side of the conditions of the aforementioned theorems.

The dynamics of a jet engine compressor which is to be controlled using a self-triggered control strategy is [26]:

$$\dot{x}_1 = -x_2 - 1.5x_1^2 - 0.5x_1^3 \tag{12a}$$

$$\dot{x}_2 = x_1 - u \ , \tag{12b}$$

where x_1 represents the mass flow, x_2 is the pressure rise, and u is the throttle mass flow, which is the input. If states x_1 and x_2 could be monitored continuously, then it is known [26] that control law

$$u = \frac{0.75x_1^6 - 1.5x_1^5 + x_1^4(2.5x_2 - 3) + x_1^3(3x_2 - 3.5)}{x_1^2 + 1} + \frac{2.25x_1^2 + x_1(2x_2^2 - 3x_2 + 0.5) + 1.5x_2}{x_1^2 + 1} \tag{13}$$

globally asymptotically stabilizes the system. It is also indicated that the *continuously-controlled* closed loop system admits a Lyapunov function

$$V(x) = 1.46\,x_1^2 - 0.35\,x_1\left(\frac{x_1^3 + 3x_1^2 - x_1 + 2x_2}{x_1^2 + 1}\right) + 1.16\left(\frac{x_1^3 + 3x_1^2 - x_1 + 2x_2}{x_1^2 + 1}\right)^2 \tag{14}$$

The self-triggered strategy of [26] suggests that instead of interrogating the sensors monitoring x_1 and x_2 at the fastest rate possible, one can instead sample them at specific time intervals only, and still stabilize the system by applying piecewise constant inputs based on the most current state measurements. The system is switching, since the control input applied during these intervals is piecewise constant. Thus we write

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_2 - 1.5x_1^2 - 0.5x_1^3\\ x_1 - u\big(x_1(\tau_i), x_2(\tau_i)\big)\end{bmatrix} = f_{\sigma(t)}(x) \ ,$$

and we assume for this example that $\tau_D = 10^{-3}$ seconds and $T_D = 1$ second. The time intervals between successive samplings, called *inter-execution times* are explicitly given in [26, (IV.29)], and adapted to the notation used in this paper. According to [26], the switching instants should satisfy

$$\Delta\tau_i \triangleq \frac{29\, x_1(\tau_i) + d\big(x(\tau_i)\big)}{5.36\, d\big(x(\tau_i)\big)\, x_1(\tau_i)^2 + d\big(x(\tau_i)\big)} \ , \qquad (15)$$

where $\Delta\tau_i \triangleq \tau_{i+1} - \tau_i$, and $d(x) = \sqrt{x_1^2 + (\frac{x_1^3+3x_1^2-x_1+2x_2}{x_1^2+1})^2}$. In [26] it is shown that the self-triggered strategy preserves the stability and performance of the continuous-time controller without requiring continuous monitoring of the system state. If a self-triggered policy is implemented, with $e(t) \triangleq x(t) - x(\tau_i)$ denoting the measurement error, then the following bound on the function in (14) can be established:

$$\dot{V} \leq -0.74 \cdot 10^8 \|x\|^4 + 0.90 \cdot 10^8 \|x\|^2\, \|e\|^2 \ . \qquad (16)$$

In [26], ISS arguments are used to establish that the self-triggered policy still ensures the monotonic decrease of the Lyapunov function, by selecting inter-execution times that do not allow e to grow enough to make the right-hand side of (16) to become positive. The following sections show how this result can be verified through the use of Theorem 1, stability can still be preserved when these inter-execution time deadlines are missed, and in addition, robustness to sustained state-update delays can be also established formally using Theorem 2.

3.1. Asymptotic stability

Assume now that due to some communication problem, sensor measurements after $t = 2$ seconds may be intermittently reported to the controller with an (additional) time delay of one inter-execution time period, i.e., the controller receives a new state update after twice the time the theory dictates. We will use Theorem 1 to show that even if the sign-definiteness of (16) is no longer guaranteed, the stability of the closed loop system is not affected.

According to [26], the measurement error e affords the following differential inequality $\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \leq 2\|e\|(\|H\|\,\|x\| + \|G\|\,\|e\|)$, where H and G are matrices which are functions of x and e, defined in [26]. In [26], these matrices are maximized within the domain $\Omega_x \triangleq \{x \mid V(x) \leq 27.04\}$; here, for convenience, we define $\mathcal{D}$ to be the largest ball contained in Ω_x, that is, $\mathcal{D} \triangleq \{x \in \mathbb{R}^2 \mid \|x\| \leq 2.34\}$.

Based on (16), one can verify that V cannot increase in $\mathcal{D}$ for $\|e\| \leq 4.6$, making Ω_x positively invariant for $\mathcal{U} = \{e \mid \|e\| \leq 4.6\}$.

With $\|H\|$ and $\|G\|$ bounded in $\mathcal{D}$, we can write $\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2 \leq \|e\| \max\{2 \cdot 2.34\,\|H\|, 2\|G\|\}(1 + \|e\|)$. Using the comparison lemma in conjunction with the fact that at the end of an inter-execution interval we are guaranteed $0.90\|e\|^2 < 0.74 \cdot 0.33^2 \|x\|$ (see [26]), we obtain $\|e(t)\| \leq 1 - \exp\left(\frac{0.5307\, t}{\tau_{i-1} - \tau_i}\right)$. Thus, after a time interval which is twice as long as the time-triggered policy of [26] suggests, the measurement error cannot be bigger than $\|e\|_{\mathsf{max}} = 3.575$. Armed with this knowledge, we can return to (16), and write

$$\begin{aligned} \dot{V} &\leq -0.74 \cdot 10^8 \|x\|^4 + 0.90 \cdot 3.575 \cdot 10^8 \|x\|^2 \Rightarrow \\ &\leq -10^6\, V^2 + 7.15 \cdot 10^9\, V \ , \end{aligned} \qquad (17)$$

where the over- and under-approximations of $\|x\|$ by V are obtained numerically, using extremum seeking methods. Invoking the comparison lemma again, we get

$$V(t) \leq \frac{b\,V(x(\tau_i))}{b\,\mathrm{e}^{-bt} + a[\mathrm{e}^{-bt} - 1]V(x(\tau_i))} \tag{18}$$

where $a < 0$ and $b > 0$ are constants. Interestingly, using $0.90\|e\|^2 < 0.74 \cdot 0.33^2 \|x\|$ in (16), we find that a faultless self-triggered control policy following the delay, will need just $2.322 \cdot 10^{-5}$ seconds to counter the increase in $V(t)$ during the delay and ensure $V(\tau_{i+1}) \leq 0.77\,V(\tau_i)$.

Now we can constructively show how the conditions of Theorem 1 are satisfied. Starting with (3), and assuming that $2\max\Delta\tau + 2.322 \cdot 10^{-5} \leq T_D$, we find that

$$\begin{aligned} V\big(x(\tau_{i+1})\big) - V\big(x(\tau_i)\big) &\leq -0.33 \cdot V\big(x(\tau_i)\big) \\ &\leq -24.42\,\|x\| \ . \end{aligned}$$

For (2) we can bound the right-hand side of (17) by $4.17 \cdot 10^{10}\,V(t)$ and then use (18) to arrive at

$$\sup_{t\in(\tau_i,\tau_{i+1}]} \dot{V} \leq \frac{2.19 \cdot 10^{14}\,T_D V(x(\tau_i))}{\tau_{i+1} - \tau_i} \times \Big[7.15 \cdot 10^3 \mathrm{e}^{-7.15\cdot 10^9 T_D} + (1 - \mathrm{e}^{-7.15\cdot 10^9 T_D})V(x(\tau_i))\Big]^{-1}$$

It follows from Theorem 1 that the system maintains its stability properties if state update is delayed by one inter-execution interval, as long as the delay is not sustained, and the system resumes its self-triggered policy after the state update. An illustration of the effect of such a delay after $t = 2$ seconds is shown in Fig. 4.

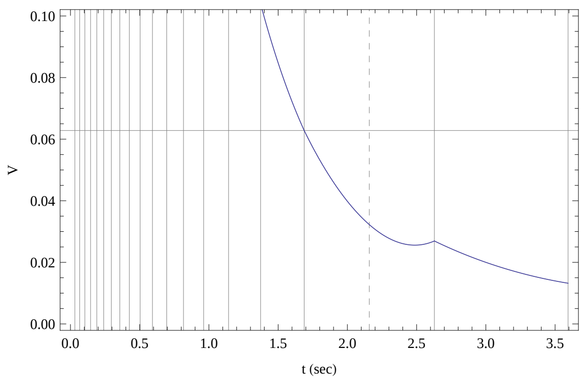

Fig. 4. The effect of a delay in the state update under a self-triggered control regime. The vertical solid lines mark the inter-execution times. The sensor fails to inform the controller about the state of the system at the time instant marked by the dashed line which was the intended inter-execution time. Instead, the sensor takes twice that time to report the state measurement. Because of this delay, the value of the V increases temporarily, but not sufficiently so that it surpasses the value it had the last time the state was updated—marked by the vertical line over 0.06. Once the communication is established again between sensor and controller, V decreases monotonically.

3.1.1. Input-to-state stability

In [26] the self-triggered system was shown to be robust—in an ISS sense—with respect to stochastic measurement noise and impulse input disturbances. Here we will demonstrate with the help of Theorem 2 that it is also robust with respect to sustained inter-execution delays.

Consider the closed loop system (12)–(13), with (13) implemented through the self-triggered policy of inter-execution times suggested by (15). Assume now that due to some time of persistent communication problem, all inter-execution times after the first second of operation are prolonged by 15%. The result of this delay is an increase of the measurement error beyond the point at which $0.90\|e\|^2 \leq 0.74 \cdot 0.33^2\,\|x\|^2$ that the self-triggered policy targeted.

Note that using the same bounds computed in Section 3.1,

$$\log(1+0.045\|x\|^2) \leq 0.045\,\|x\|^2 \leq V(x) \leq \sqrt{74}\,\|x\|^2 \ ,$$

where it can be seen that $\log(1+0.045\|x\|^2)$ is Lipschitz. Starting from (16), and with some manipulation, we arrive at

$$\sup_{t\in(\tau_i,\tau_{i+1}]} \tfrac{\partial V}{\partial x}\left(f_{\sigma(t)}(x)+u\right) \leq \tfrac{0.45\cdot 10^8}{\tau_{i+1}-\tau_i}(V+\|e\|_\infty)^2$$

which establishes (6). For (7), recalling (16) which holds during a regular inter-execution time interval, we can use the upper and lower bounds on V to obtain

$$\dot{V} \leq -89.05\cdot 10^4\,V^2 \Rightarrow V(t) \leq \frac{V(\tau_i)}{1+89.05\cdot 10^4\,V(\tau_i)},$$

for $t\in(\tau_i,\tau_i+\Delta\tau_i]$. After $\tau_i+\Delta\tau_i$ we can still have $\dot{V} \leq 0.90\cdot 10^8\,\|e\|_\infty V$, on which the comparison lemma applies with the above bound in the role of initial condition, to yield

$$V(\tau_{i+1}) - V(\tau_i) \leq -V(\tau_i) + c_1\,\mathrm{e}^{c_2\,\|e\|_\infty} \ ,$$

where $c_1 = 7.15\cdot 10^3$ and $c_2 = 1.35\cdot 10^7$. Thus, (7) is established.

Figure 5 shows the actual effect of a persistent 15% delay in state updates on top of the inter-execution time. The latency starts occurring after $t=1$ seconds.

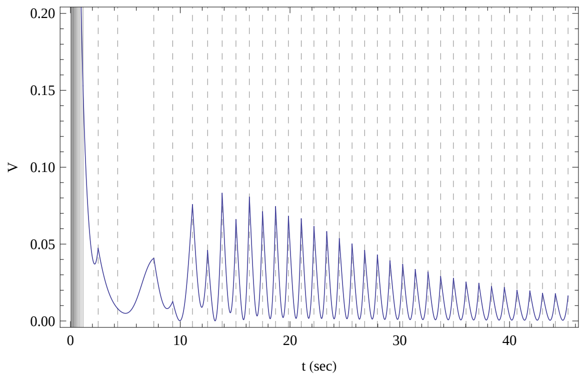

Fig. 5. The effect of a consistently 15% longer inter-execution interval in a self-triggered control regime. The vertical solid lines mark the regular inter-execution times, and the dashed ones indicate the delayed ones. During the first part of the inter-execution interval, the function decreases as predicted, but the prolonged time between state updates causes an eventual increase toward the end of the interval. The amplification is remains bounded and the maximum values of the function between state updates reduce with time but do not converge to zero.

IV. Conclusion

There can be classes of stable systems switching with dwell time, or not switching at all, for which existing uniform asymptotic stability conditions are violated. These systems can still be shown to be asymptotically stable, for a selected switching signals; this entails the relaxation of the conditions imposed on the derivative of some (common) Lyapunov-like function. Essentially one just needs to ensure that the function decreases measurable by the end of each switching interval. This type of stability conditions are difficult to establish analytically, however, in practical examples of interest they can either be guaranteed by design, or found to hold numerically. The stability relaxation presented in this paper also admits an input-to-state stability extension.

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