Probability of success in stochastic robot navigation with state feedback

Shridhar K. Shah, Chetan D. Pahlajani and Herbert G. Tanner

Abstract— **The analysis in this paper applies to robots with dynamics described by a stochastic differential equation, which need to navigate in constrained environments. The approach offers a method to calculate the probability that a feedback control policy designed for the drift component of the dynamics, will succeed in allowing the robot to avoid collisions and converge to its navigation goal in the presence of stochastic (white) noise. The problem is formulated as an exit problem and known techniques in the field of stochastic processes are brought to bear to determine the probabilities that the stochastic process describing the motion of the robot will "exit" the workspace through a particular part of the boundary. We motivate the use of this analysis using a controller constructed using negative gradient of a navigation function and give the analytic solution for the case of a constrained but obstacle-free workspace.**

Keywords - stochastic differential equations, uncertainty, probability, exit time

I. INTRODUCTION

There are practical cases of interest where robots are deployed in real-life environments and are subject to disturbances of a stochastic nature. Wind gusts, water currents or unstructured terrain, for example, can sometimes be reasonably modeled only as stochastic processes. Under such type of uncertainty, a robotic system can be represented as a stochastic process, the evolution of which is described by a stochastic differential equation.

Lyapunov analysis tools for deterministic systems have been extended to the stochastic domain. However, stability and convergence results are either too strict to establish, or inadequate for a problem like robot navigation. For example, to establish almost sure convergence [1] (convergence to some equilibrium with probability one) one might need unbounded inputs, or impose unrealistic assumptions on the structure of the diffusion term in the stochastic differential equation; on the other hand, even if convergence with probability one is established [2] in a case of robot navigation, one may still have no guarantees that the system will not hit obstacles on its way to the equilibrium. Weaker results in the form of convergence to some invariant set or ultimate boundedness in expectation [3] are subject to similar limitations.

Instead of searching for a controller that would give almost sure convergence under unrealistic assumptions on the diffusion term of stochastic differential equation, one may wish to work with a controller that can provably stabilize the drift part of the stochastic dynamics, and obtain some formal guarantees on the expected performance of that controller when implemented on a stochastic system with the same drift. Such formal performance guarantees can be obtained in the form of the probability of the robot hitting a particular section of its workspace boundary; if that section surrounds the navigation goal, this number gives the probability of success. In the context of stochastic processes, this problem falls into the category of *exit problems*, which focus on Markov processes evolving in bounded regions and are concerned with the location at which a sample path will exit the region and the time at which this is likely to happen (exit time). Solution methods have a long history, and different formulations have been developed, both in discrete and continuous time and space, for different types of applications ranging from finance to biology and control.

The paper thus suggests an exit location problem formulation as an analysis and performance evaluation tool for robots with stochastic dynamics that have to move in a constrained environment. Related work exists in the context of *optimal exit time control* problems, although the objective may be quite different. In one particular formulation [4], [5], one searches for an optimal control that is supposed to keep the state of a continuous-time stochastic differential equation inside a region for up to a predefined expected exit time. Another problem formulation in the same context [6] focuses on finding an optimal control to maximize probability of hitting a particular part of the boundary, is developed for discrete time Markov control processes, and attacks the problem of maximizing the probability of reaching a target set before hitting a non-desired set (cemetery) using dynamic programming. When the objective is to find a control law that minimizes a cost while satisfying a probabilistic constraint, then the problem is referred to as chance-constrained optimization. Only convex problems of this type have been solved efficiently [7], [8], and the suitability of such an approach on computationally constrained systems is questionable.

All these formulations develop around an exit problem for a stochastic dynamical system with inputs. The approach in [4] is similar to the one in this paper in the sense that it relies on the solution of a PDE. The control laws derived in the approaches of [4], [5], however, minimize a spatial integral of the norm of the control input, so they can be thought of as giving an answer to the problem of how to maintain the invariance of a region for up to a given time with the most "inexpensive" input. The formulation in [6], on the other hand, would be directly applicable here had it not been developed for discrete time problems. The main concern

This work is supported by ARL MAST CTA # W911NF-08-2-0004

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here however is the on-line computational overhead, which is prohibitive for the applications envisioned in this work, involving implementations on miniature robotic platforms with very limited computational capabilities. Indicative of the level of computational complexity involved, the derivation of the optimal control law in [4] requires the solution of two partial differential equations at each time step.

Not being able to afford such solutions, the approach in this paper circumvents the optimal control design problem by first suggesting a stabilizing controller (not necessarily optimal) for the stochastic robot navigation system, which can be implemented when no constraints on the size of the control input are imposed. (Same caveats as in cases in existing literature apply.) Next, a method for computing the probabilities of success (and failure) in the navigation problem for the given control law is presented. The method also involves the solution of a partial differential equation, but since the process is intended for verification and performance analysis, it is reasonable to assume that it can be performed off-line. It is indicated that in the special case of a constrained spherical environment with no internal obstacles, the symmetry of the problem admits an analytic solution of the partial differential equation.

Section II sets the problem formulation while section III suggests a stabilizing controller for the stochastic dynamical system, in cases where there is no *a priori* controller to test the performance of. In section IV we present the main result which deals with the case of finding the probability of the sample path exiting through a particular section of the boundary. Section V presents a simple example where analytical solutions can be obtained, and is followed by section VII which summarizes the paper and outlines ongoing research directions.

II. Problem Setup

In the scope of this work are robotic systems modeled with stochastic differential equations. In these equations, the control input appears as drift, and the stochastic noise component enters as the diffusion term. If such a system were not subject to the stochastic perturbation, its dynamics would be written as

$$\dot{x} = u(x,t), \tag{1}$$

where $x \in \mathcal{W} \subseteq \mathbb{R}^n$ is the state and $u \in \mathbb{R}^n$ is the control input. The function $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is assumed to be C^2 and locally Lipschitz. The robot (1) is assumed to move in a workspace $\mathcal{W}$ which is a bounded, open subset of $\mathbb{R}^n$. The closed set $\mathcal{O} \subset \mathcal{W}$ represents obstacles (forbidden regions) in the robot's workspace. The free workspace is thus $\mathcal{P} \triangleq \mathcal{W} \setminus \mathcal{O}$.

Under the assumption of the dynamics being given in the form of a single integrator, and the obstacle and workspace sets have spherical outer boundaries (or at least, star-shaped [9]), it can be shown, that the negated gradient of a particular function called a *navigation function* [9]

$$\varphi(x) = \left(\frac{\|x\|^{2k}}{\|x\|^{2k} + \beta(x)} \right)^{\frac{1}{k}}, \tag{2}$$

can be used in the place of u in (1) to yield asymptotic stability for the the closed loop dynamics from almost all initial conditions and ensure obstacle avoidance, for a parameter k sufficiently big. In (2), the function $\beta : \mathcal{P} \to [0,\infty)$ is defined as

$$\beta \triangleq \prod_{j=0}^{M} \beta_j,$$

with

$$\beta_0 \triangleq \rho_0^2 - \|x\|^2$$
$$\beta_j \triangleq \|x - x_j\|^2 - \rho_j^2, \quad j = 1, \ldots, M.$$

With these expressions for β_j, the robot's workspace is taken to be a ball of radius ρ_0, which contains M spherical obstacles with radii ρ_j and centers x_j, $j = 1, 2, \ldots, M$, $k \in \mathbb{N}$ is a sufficiently large positive integer and $\|x\|$ is the Euclidean norm on $\mathbb{R}^n$. It is also assumed that the obstacles are isolated: the regions defined as the sets of states where some β_j is negative, do not overlap; then the workspace is said to be *valid*. In the case of a valid workspace, the controller $u(x) \triangleq -\nabla\varphi(x)$ yields positive invariance for the level sets of φ and gives asymptotic convergence of (1) to the origin (which is assumed to be the goal configuration). The robustness properties of such a system under stochastic disturbance have not been studied, however.

In the presence of stochastic disturbance, the system (1) should be modeled by a stochastic differential equation. To construct such a stochastic model, let $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and define $b : \mathbb{R}^n \to \mathbb{R}^n$ with $b(x) = -\nabla\varphi(x)$ for $x \in \mathcal{P}$ such that the Lipschitz and linear growth conditions hold:

$$\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \le K\|x - y\|,$$
$$\|b(x)\|^2 + \|\sigma(x)\|^2 \le K^2(1 + \|x\|^2),$$

for $x, y \in \mathbb{R}^n$, and K a positive constant. For the $n \times m$ matrix σ, the norm $\|\sigma\|$ is defined as

$$\|\sigma\|^2 \triangleq \sum_{i=1}^{n}\sum_{j=1}^{m} \sigma_{ij}^2.$$

Let $W = \{W_t, \mathcal{F}_t : 0 \le t < \infty\}$ be an m-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the sample space, $\mathcal{F}$ is a σ-algebra on Ω, $\mathbb{P}$ is the probability measure and $\{\mathcal{F}_t : t \ge 0\}$ is the filtration (i.e. an increasing family of sub-σ-algebras of $\mathcal{F}$) that satisfies the usual conditions.[1] The stochastic dynamics of the robot can be described by

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t, \\ X_0 &= x, \end{aligned} \tag{3}$$

with initial condition $x \in \mathcal{P}$. The global Lipschitz and linear growth conditions ensure [10, Theorems 5.2.5 and 5.2.9] that the stochastic differential equation (3) has a globally (in time) defined strong solution $\{X_t, \mathcal{F}_t : 0 \le t < \infty\}$ which is

[1]The filtration is said to satisfy usual conditions if it is right continuous and $\mathcal{F}_0$ contains all P-null sets [1].

square-integrable. Also, strong uniqueness holds for the pair of coefficients (b, σ).

While the deterministic system $\dot{x} = -\nabla\varphi(x)$ is guaranteed to avoid obstacles (and the workspace boundary) and converge to the origin as $t \to \infty$ for almost all initial conditions, the same cannot be said of the stochastic system (3). Indeed, under some simple non-degeneracy conditions on the noise, the process X_t exits the free workspace $\mathcal{P}$ (i.e. hits the boundary or an obstacle) in finite time *with probability one*. It is thus of interest to know the probability of X_t reaching a sufficiently small neighborhood of the destination (origin) before exiting $\mathcal{P}$.

III. A Stabilizing Controller

One straightforward extension of a deterministic control design strategy based on the navigation function construction of (2) to the case where the robotic system experiences stochastic disturbances and is governed by dynamics of the form (3) would yield a closed loop dynamics of the form

$$\begin{aligned} dX_t &= -\nabla\phi(X_t)\; u\; dt + \sigma(X_t)dW_t \qquad (4)\\ X_0 &= x, \end{aligned}$$

where now the objective is to find a function u that appropriately *modifies* the deterministic control strategy to yield almost sure convergence. One solution is to apply the universal formula of [11] and construct the modifying function u as follows.

$$u(X_t) = \xi \cdot \psi\left(\xi\right),$$

where, ψ is a smooth function satisfying

$$\psi(\xi) = \begin{cases} 0 & \text{if } \xi \le 0\\ 1 & \text{if } \xi \ge 1\\ 0 < \psi(\xi) < 1 & \text{otherwise,} \end{cases}$$

$$\frac{\partial\psi}{\partial\xi} \ge 0,$$

and its argument is $\xi = -\frac{c_v(X_t)}{b_v(X_t)} + 1$, where

$$\begin{aligned} b_v(X_t) &= -\|\nabla\phi\|^2\\ c_v(X_t) &= \frac{1}{2}\sum_{i,j=1}^{n} \sigma^T\sigma\frac{\partial^2\phi}{\partial x_i \partial x_j}. \end{aligned}$$

One possible suggestion for a function ψ that can satisfy the requirements stated above is given in [12, Theorem 5.1].

The above controller can be shown to stabilize (4), but its values are not bounded: it is clear that $u \to \infty$ as $b_v(\cdot) \to 0$. On the other hand, in the case of bounded inputs, little can be said about the stability of the system since the stochastic perturbation produced by dW_t can potentially be unbounded, even if $\sigma(\cdot)$ is. The next section treats the case where the controller is actually given, and the question is how well can this controller be expected to perform in the presence of unbounded stochastic perturbations. One relaxation introduced is that the system will be considered to have achieved its objective not if it closes in on the destination asymptotically, but rather when it enters a sufficiently small neighborhood of it, at which point of time it is assumed to "stop" and the process terminated.

IV. Probability of Exit through a Boundary

This section describes a fairly standard formulation of exit time problems for stochastic differential equations. We demonstrate how this framework can can be used to compute the probability that a sample path X_t corresponding to a trajectory of a mobile robot subject to noise, reaches a neighborhood of its destination (assumed to be the origin) before hitting either the obstacles or the boundary of the workspace.

To this end, fix an arbitrarily small $\epsilon > 0$ so that the closed ball centered at the origin with radius ϵ, denoted $\overline{\mathcal{B}(0,\epsilon)}$, is contained in $\mathcal{P}$. Let $\mathcal{D} \triangleq \mathcal{P} \setminus \overline{\mathcal{B}(0,\epsilon)}$. The boundary of $\mathcal{D}$ can be decomposed into the disjoint union of

- $\mathcal{D}_1 \triangleq \{x \in \mathcal{P} : \|x\| = \epsilon\}$, i.e. the boundary of the closed ball $\overline{\mathcal{B}(0,\epsilon)}$, and
- $\mathcal{D}_2$ which is the finite union of the boundaries of the obstacles and the boundary of the workspace $\mathcal{P}$.

Let $\tau_{\mathcal{D}}$ be the first exit time of the process X_t from $\mathcal{D}$, i.e.

$$\tau_{\mathcal{D}} \triangleq \inf\{t \ge 0 : X_t \notin \mathcal{D}\}.$$

We are interested in $\mathbb{P}(X_{\tau_{\mathcal{D}}} \in D_1)$, that is, the probability that X_t reaches the destination before hitting an obstacle or the boundary of the workspace. The problem is thus formulated in terms of an *exit problem* for a diffusion process from a domain $\mathcal{D}$.

We start by introducing some notation. First, let $a : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be the matrix $a(x) \triangleq \sigma(x)\sigma^T(x)$. Next, let $\mathfrak{A}$ be the second-order partial differential operator

$$\mathfrak{A} \triangleq \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(x)\frac{\partial^2}{\partial x_i \partial x_k}. \qquad (5)$$

The following observation will be central to the calculations: for any $v \in C^2(\mathbb{R}^n)$ with bounded first and second order partial derivatives, Ito's formula [10] gives

$$v(X_t) = v(X_0) + \int_0^t \mathfrak{A}v(X_s)\; ds + M_t^v, \qquad (6)$$

where

$$M_t^v \triangleq \sum_{i=1}^{n}\sum_{k=1}^{m}\int_0^t \sigma_{ik}(X_s)\frac{\partial v}{\partial x_i}(X_s)dW_s^{(k)}$$

is a zero-mean martingale. Here, $W_t^{(k)}$ is a component of the m-dimensional Brownian motion W_t. Finally, we compactly indicate the initial condition in the stochastic differential equation (3) by writing $\mathbb{P}^x$ and $\mathbb{E}^x$ to denote, respectively, probabilities and expectations computed under the initial condition $X_0 = x$.

Under a fairly mild assumption on non-degeneracy of noise, the matrix a ensures that $\mathbb{E}^x[\tau_{\mathcal{D}}] < \infty$ for all $x \in \mathcal{D}$. Moreover, expectations of the form $\mathbb{E}^x[f(X_{\tau_{\mathcal{D}}})]$ for any continuous function $f : \partial\mathcal{D} \to \mathbb{R}$ (here, $\partial\mathcal{D}$ represents the boundary of $\mathcal{D}$) can be computed by finding a function $u \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ which solves the Dirichlet problem

$$\mathfrak{A}u = 0 \qquad \text{in } \mathcal{D} \tag{7a}$$
$$u = f \qquad \text{on } \partial\mathcal{D}. \tag{7b}$$

The following statement is actually a direct consequence of [10, Proposition 7.2] combined with [10, Lemma 7.4]. We give its proof primarily for completeness purposes.

Theorem 1: Suppose that for some $1 \le \ell \le n$, we have

$$\min_{x \in \overline{\mathcal{D}}} a_{\ell\ell}(x) > 0. \tag{8}$$

Then, $\mathbb{E}^x[\tau_{\mathcal{D}}] < \infty$ for all $x \in \mathcal{D}$. If $u \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ is a solution to the Dirichlet problem (7) with $f : \partial\mathcal{D} \to \mathbb{R}$ a continuous function, then

$$u(x) = \mathbb{E}^x[f(X_{\tau_{\mathcal{D}}})].$$

In particular, if $f = \chi_{D_1}$ (the indicator function of D_1),[1] then

$$u(x) = \mathbb{P}^x(X_{\tau_{\mathcal{D}}} \in \mathcal{D}_1). \tag{9}$$

Proof: We first establish that (8) implies $\mathbb{E}^x[\tau_{\mathcal{D}}] < \infty$ for all $x \in \mathcal{D}$. Let

$$b \triangleq \max_{x \in \overline{\mathcal{D}}} \|b(x)\|, \quad a \triangleq \min_{x \in \overline{\mathcal{D}}} a_{\ell\ell}(x), \quad q \triangleq \min_{x \in \overline{\mathcal{D}}} x_\ell.$$

Fix $\nu > (2b/a)$, and consider a function $h \in C^2(\mathbb{R}^n)$ with bounded first and second partial derivatives such that $h(x) = -\mu\, e^{\nu x_\ell}$ on some open set containing $\overline{\mathcal{D}}$, where the constant $\mu > 0$ will be chosen later. Then, for points $x \in \overline{\mathcal{D}}$, it is seen that

$$\begin{aligned} -\mathfrak{A}h(x) &= \frac{1}{2}\mu\,\nu e^{\nu x_\ell} a_{\ell\ell}(x)\left(\nu + \frac{2b_\ell(x)}{a_{\ell\ell}(x)}\right) \\ &\ge \frac{1}{2}\mu\nu\, e^{\nu q} a\left(\nu - \frac{2b}{a}\right) \\ &\ge 1, \end{aligned}$$

for $\mu > 0$ sufficiently large. By (6),

$$h(X_t) = h(X_0) + \int_0^t \mathfrak{A}h(X_s)ds + M_t^h,$$

and hence

$$\begin{aligned} h(X_{t\wedge\tau_{\mathcal{D}}}) &= h(X_0) + \int_0^{t\wedge\tau_{\mathcal{D}}} \mathfrak{A}h(X_s)ds + M^h_{t\wedge\tau_{\mathcal{D}}} \\ &\le h(X_0) - (t \wedge \tau_{\mathcal{D}}) + M^h_{t\wedge\tau_{\mathcal{D}}}, \end{aligned}$$

where $t \wedge \tau_{\mathcal{D}} \triangleq \min(t, \tau_{\mathcal{D}})$. It can be shown that $M^h_{t\wedge\tau_{\mathcal{D}}}$ is a zero-mean martingale, we take expectations and rearrange to get

$$\begin{aligned} \mathbb{E}^x[t \wedge \tau_{\mathcal{D}}] &\le h(x) - \mathbb{E}^x[h(X_{t\wedge\tau_{\mathcal{D}}})] \\ &\le 2\max_{y \in \overline{\mathcal{D}}} |h(y)| < \infty. \end{aligned}$$

Letting $t \to \infty$, we get $\mathbb{E}^x[\tau_{\mathcal{D}}] < \infty$.

Now suppose that $u \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ is a solution to the Dirichlet problem (7). For $n \in \mathbb{N}$, let $\mathcal{D}_n \triangleq \{x \in \mathcal{D} : \inf_{y \in \partial\mathcal{D}} \|x - y\| > 1/n\}$. Then $\{\mathcal{D}_n\}_{n=1}^\infty$ is an increasing sequence of open subsets of $\mathcal{D}$ whose union is $\mathcal{D}$. For $n \in \mathbb{N}$, let $\tau_{\mathcal{D}_n} \triangleq \inf\{t \ge 0 : X_t \notin \mathcal{D}_n\}$. Let $u_n \in C^2(\mathbb{R}^n)$ with bounded first and second order partial derivatives such that $u = u_n$ on some open set containing $\mathcal{D}_n$. Then, by (6), we have

$$u_n(X_t) = u_n(X_0) + \int_0^t \mathfrak{A}u_n(X_s)ds + M_t^{u_n}$$

and hence

$$u_n(X_{t\wedge\tau_{\mathcal{D}_n}}) = u_n(X_0) + \int_0^{t\wedge\tau_{\mathcal{D}_n}} \mathfrak{A}u_n(X_s)ds + M^{u_n}_{t\wedge\tau_{\mathcal{D}_n}}$$

Noting that $u_n = u$ on some open set containing $\mathcal{D}_n$, $\mathfrak{A}u = 0$ in $\mathcal{D}$ and that $M^{u_n}_{t\wedge\tau_{\mathcal{D}_n}}$ is a zero-mean martingale, we take expectations to get

$$\mathbb{E}^x u(X_{t\wedge\tau_{\mathcal{D}_n}}) = u(x). \tag{10}$$

Since $\mathbb{E}^x[\tau_{\mathcal{D}}] < \infty$, we have $\tau_{\mathcal{D}} < \infty$ with probability one. It is not hard to argue that on the set $\{\tau_{\mathcal{D}} < \infty\}$ (which has probability one), $\tau_{\mathcal{D}_n} \nearrow \tau_{\mathcal{D}}$. Now, since X_t has continuous sample paths, it follows that $X_{t\wedge\tau_{\mathcal{D}_n}} \to X_{t\wedge\tau_{\mathcal{D}}}$, $\mathbb{P}^x$-almost surely. Recalling that $u \in C(\overline{\mathcal{D}})$ is bounded, taking limits as $n \to \infty$ in (10), we get by the bounded convergence theorem

$$\mathbb{E}^x u(X_{t\wedge\tau_{\mathcal{D}}}) = u(x).$$

Now letting $t \to \infty$, the bounded convergence theorem (again) together with the fact that $u = f$ on $\partial\mathcal{D}$ gives the desired result. ■

Remark 2: Under the assumption that $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, the indicator function χ_{D_1} is continuous on the boundary. The function is defined as, $\chi_{\mathcal{D}_1} : \{\mathcal{D}_1 \cup \mathcal{D}_2\} \to \{0, 1\}$, where

$$\chi_{\mathcal{D}_1}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{D}_1, \\ 0 & \text{if } x \in \mathcal{D}_2. \end{cases}$$

The function $\chi_{\mathcal{D}_1}$ is continuous at any point $c \in \{\mathcal{D}_1 \cup \mathcal{D}_2\}$ because if $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, then, that for every $\epsilon > 0$, $\exists\delta > 0$ such that $\forall x \in \{\mathcal{D}_1 \cup \mathcal{D}_2\}$,

$$|x - c| < \delta \Longrightarrow |\chi_{D_1}(x) - \chi_{D_1}(c)| < \epsilon.$$

V. Example

To see how the results of the previous sections apply, consider a point robot in a bounded, two-dimensional obstacle-free workspace. The feedback law for this system is given in the form of the negative gradient of a navigation function, which stabilizes it in the case where $\sigma \equiv 0$. When $\sigma \neq 0$, however, neither convergence nor obstacle avoidance can be guaranteed for the dynamics.

Assume that the position of the robot is prescribed by the coordinates $\mathbf{x} \triangleq (x, y)$ and let its dynamics be of the form

$$dx = -K\frac{\partial\phi}{\partial x}u\; dt + \sigma_x d\mathcal{W}_t^1 \tag{11}$$
$$dy = -K\frac{\partial\phi}{\partial y}u\; dt + \sigma_y d\mathcal{W}_t^2 \tag{12}$$

where, $\sigma_x = \sigma_y = 1$ on $\mathbf{x} \in \mathbb{R}^2\backslash\{(0,0)\}$ and $\sigma_x \equiv \sigma_y \equiv 0$ for $\mathbf{x} = (0,0)$, ϕ is the navigation function, K is the gain

[1] The indicator function can be shown continuous on the boundary, see remark 2.

($K \geq 1$, $K \in \mathbb{N}$) and, u is assumed to be a "modifying" function, which we will later use to compensate for diffusion term.

Consider workspace of radius R and a small ball of radius ε close to origin.

$$\mathcal{B}(0, R) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < R\};$$
$$\mathcal{B}(0, \varepsilon) = \{x \in \mathbb{R}^2 : \|\mathbf{x}\| < \varepsilon\};$$
$$\mathbf{x} = (x, y) \in \mathcal{B}(0, R), \quad \|\mathbf{x}\| = \sqrt{x^2 + y^2}.$$

For simplicity assume $R = 1$, Then, a trivial navigation function can be given as

$$\phi(\mathbf{x}) = x^2 + y^2, \quad \frac{\partial \phi}{\partial x} = 2x, \quad \frac{\partial \phi}{\partial y} = 2y.$$

A. Stabilizing controller

Using method presented in section III, we can find a control law for the presented system (11)-(12).

For unity gain $K = 1$, the system can be written as,

$$dx = -\frac{x}{x^2 + y^2} dt + d\mathcal{W}_t^1,$$
$$dy = -\frac{y}{x^2 + y^2} dt + d\mathcal{W}_t^2,$$

If we use $V = \phi = x^2 + y^2$ as a Lyapunov function candidate, then according to [1]

$$\mathcal{L}V = -1 < 0,$$

where, $\mathcal{L}$ denotes the (stochastic) Lie directional derivative of a function (in this case, same as $\mathfrak{A}$ defined in (5)). This implies that the stochastic system is asymptotically stable with probability one [11]. However, it should be noted here that the drift term which functions as the controller for the robot becomes unbounded as $x^2 + y^2 \to 0$.

B. Probability of reaching the neighborhood of the origin

The method presented in this section allows to analyze the performance of the controller without saturating the input or assuming it to be unbounded. To see what would be the effect on the stochastic dynamics of a controller designed based on the unperturbed nominal system, take $u = 1$. Then the partial differential equation that is used to compute the probability of reaching a small neighborhood of the origin can be given according to (7) as

$$-2Kx\frac{\partial u}{\partial x} - 2Ky\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial y^2} = 0$$
$$u\mid_{\partial \mathcal{B}(0,\varepsilon)} = a, \quad u\mid_{\partial \mathcal{B}(0,1)} = b.$$

In this particular example, due to the symmetry of the problem, it is easier to express and solve the partial differential equation in polar coordinates: Assuming $u(x, y) = f(r, \theta)$, the partial differential equation can be represented as

$$\mathcal{L}f(r, \theta) = \frac{1}{4K}\frac{\partial^2 f}{\partial r^2} + \left(\frac{1}{4Kr} - r\right)\frac{\partial f}{\partial r} + \frac{1}{4Kr^2}\frac{\partial^2 f}{\partial \theta^2} = 0,$$

where $\varepsilon < r < 1$. The boundary conditions are

$$f(\varepsilon, \theta) = a, \quad f(1, \theta) = b, \quad \text{for } a \neq b.$$

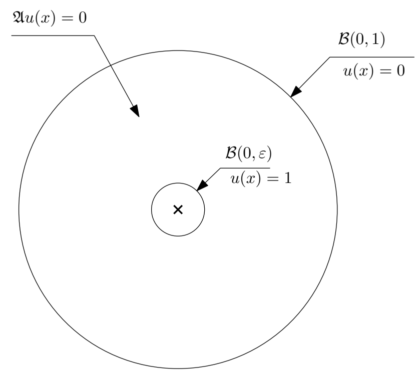

Fig. 1. The figure shows the problem setup explained above, with $a = 1$ and $b = 0$. Solution of the PDE, $\mathfrak{A}u = 0$ gives the probability of reaching a boundary close to origin before hitting the other.

Because of symmetry, the partial differential equation reduces to an ordinary differential equation. Letting $f(r, \theta) = R(r)$ yields

$$\frac{1}{4K}R''(r) + \left(\frac{1}{4Kr} - r\right)R'(r) = 0.$$

Then the solution of this ordinary differential equation can be given as

$$f(r, \theta) = R(r) = a + C_1 \int_{\varepsilon}^{r} \frac{e^{2Ks^2}}{s} ds, \qquad (13)$$

where

$$C_1 = \frac{b - a}{\int_{\varepsilon}^{1} \frac{e^{2Ks^2}}{s} ds}.$$

To find the probability of reaching a set arbitrarily close to origin, let us assume $\varepsilon = 0.01$, and that the system (11)-(12) is initiated within a ball of radius 0.5 around the origin. The function f has to be choosen to be an indicator function according to (9), with $a = 1$ and $b = 0$. Then, the probability that the system reaches the goal set before hitting the obstacle boundary is found to be $\mathbb{P}^{(0.5,\theta)}(X_{\tau_{\mathcal{D}}} \in \mathcal{D}_1) = 0.349$, for $K = 1$ using numerical approximation of integrals appearing in (13).

It should be noted that this probability increases with the control gain K:

$$K = 2: \qquad \mathbb{P}^{(0.5,\theta)}(X_{\tau_{\mathcal{D}}} \in \mathcal{D}_1) = 0.6599$$
$$K = 3: \qquad \mathbb{P}^{(0.5,\theta)}(X_{\tau_{\mathcal{D}}} \in \mathcal{D}_1) = 0.8907.$$

VI. Simulation Results

We performed simulations of the system (11)-(12), with $\phi(\mathbf{x}) = x^2 + y^2$ and rest of the parameters according to section V-B. The simulations were performed in MATLAB®. One hundred sample paths were computed for values of $K = 1, 2$ and 3. It should be noted that K is here just a scaling factor and acts as a controller gain; higher values of K uses larger inputs.

Each simulated path was checked for intersection with either of the outer or inner circles, of radius $R = 1$ and $\varepsilon = 0.01$ respectively, and stopped upon intersection. Paths that do not intersect within six simulation seconds were ignored, thought of as possibly resulting in either success or failure. With initial condition $(x, y) = (0.3, 0.4)$, and $r = 0.5$, 100 sample paths were simulated for 6 seconds for each K using $dt = 10^{-4}$. We found that 40% of sample paths touched the center circle of radius 0.01 (succeeded) for K=1, while 62% out of 100 did so for $K = 2$, and 88% for $K = 3$. The theory predicted these probabilities at the levels of 0.349, 0.6599 and 0.8907 respectively. It should be noted that for higher number of trials the accuracy of the solution converges to the theoretical value, as well as the accuracy of numerical simulations increases with smaller discretization time [13]. For example, for $K = 1$ and $dt = 10^{-5}$ seconds, the percentage of successful sample paths is indeed 35% for 100 simulations of 6 seconds.

Two sample paths are shown in figure 2. This simulations may abstractly represent the planar motion of a UAV affected by wind disturbances, or a AUV under effect of water currents. The probability of reaching the goal before hitting the restricted region for a given initial position can be computed using the presented method.

In cases with obstacles, the PDE may not be solvable analytically, but existing numerical methods suffice. In a simple, one obstacle case, the system becomes an elliptic PDE with variable coefficients. For the methods of solutions of such PDEs we refer to [14], [15].

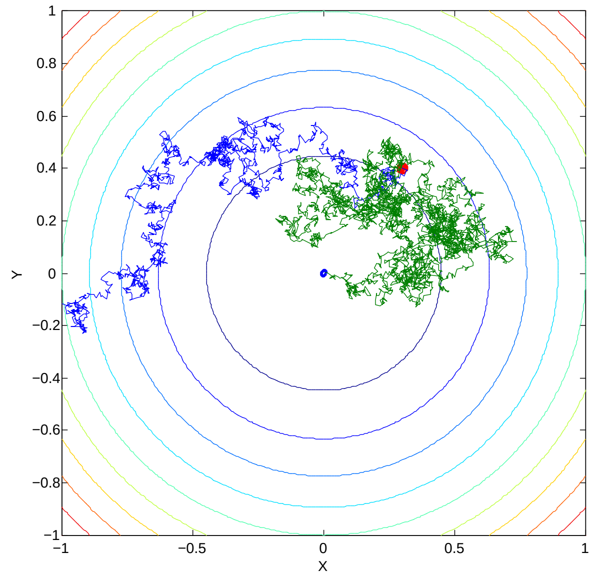

Fig. 2. The figure shows two sample paths generated using MATLAB® Econometrix toolbox. The SDE drift and diffusion terms are $-\nabla\phi$ and I (identity) respectively, for $K = 1$. The concentric circles represent the potential field generated by ϕ. Red dot at (0.3,0.4) is the initial condition of the robot. The blue and green paths exit at the boundary of radius $R = 1$ shown as cyan circle and $\varepsilon = 0.01$ shown as small blue circle respectively.

VII. Conclusions

Existing stabilization methods may not be applicable to the problem of robot navigation when the dynamics are of stochastic nature. Controller designs under the assumption of unbounded inputs can be used to establish asymptotic stability with probability one. If the assumption of unbounded inputs is unreasonable, then the possibility of disturbances with arbitrarily big magnitude makes a formal guarantee of convergence with collision avoidance elusive. Then, the performance of the system can be analyzed in terms of the probability that its controller succeeds in bringing the system within an arbitrarily small neighborhood of the target position. The analysis takes the form of an exit time problem, and solutions in general require the solution of a partial differential equation which can be performed off-line in principle. In cases where the problem has symmetries and the environment in which the robot moves is not cluttered, analytic solutions might also be possible.

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