ISS Properties of Nonholonomic Mobile Robots

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***Abstract*—The paper presents the first result on ISS properties of dynamic unicycle models describing nonholonomic mobile robots. It is known that ISS is related to smooth stabilizability, however this relation cannot exclude the possibility of non smoothly stabilizable systems to possess ISS properties. In fact, it is shown that in a certain topology that seems to suit the nonholonomic nature of the mobile robot, and by applying a particular control law, the closed loop system can be made locally ISS by an appropriate choice of control inputs. Apart from any possible theoretical ramifications, this result encourages an ISS-based stability analysis of groups of mobile robots.**

I. Introduction

Input-to-state stability (ISS) [1] is a framework for stability and robustness analysis that has proved to be extremely useful in a variety of applications, from biochemical networks [2] to formation control and robotics [3], [4]. The success of input-to-state stability as an analysis tool is due in part to its invariance properties under a large class of system interconnections [1]. It is also known that ISS is closely related to the ability to stabilize the system using smooth control inputs [5].

Lately there has been related work on applications of input-to-state stability on vehicle formations [3], [4], [6]. Taking advantage of the invariance properties of ISS, one can come up with stability measures and error bounds for the vehicles in the formation, that depend on the group leaders input [7]. This work has primarily focused on linear dynamics, such as those obtained through input-output feedback linearization. However, vehicles in general, and mobile robots in particular, are generally described in terms of nonlinear, nonholonomic models. It is thus natural to ask if such techniques can be applicable to the case where complete nonholonomic dynamics are concerned.

The highly nonlinear nature of the problem and the peculiarity of the nonholonomic dynamics make the problem particularly challenging. Moreover, the known result that relates ISS to smooth stabilization may tempt one to think that systems that cannot be smoothly stabilized may not possess any ISS properties. Although in general this may indeed be the case, in this paper we show that in a certain topology which seems to characterize better the nonholonomic nature of the system, it is possible to establish local ISS properties. Beyond any theoretical implications regarding the nature of the nonholonomic dynamics, and the insight that one can gain through an ISS analysis, this result seems to be implying that generalization of ISS-based formation control to more detailed, nonlinear vehicle models may be possible.

The result presented in this paper is primarily based on the introduction of a particular metric on the state space of the nonholonomic robot and the use of a discontinuous, nonholonomic feedback controller. The stability of the closed loop system is established through a (nested) singular perturbation analysis that imposes certain conditions on the controller gains. It is also worth noting, that the nonlinear gain estimate from disturbances to state can be expressed as a linear function of the magnitude of the disturbances. This is important because it opens the way to efficient gain computation algorithms that can scale easily in large groups of mobile robots.

The rest of the paper is organized as follows: in section II we give a brief description of the problem at hand and we introduce the metric. Section III introduces the dynamic nonholonomic feedback controller and establishes its stability properties using singular perturbations arguments. In section IV the closed loop system is shown to be locally ISS with respect to acceleration input disturbances. Section V verifies the theoretical results via numerical simulations. Finally, section VI concludes the paper with a summary of the contributions of this paper.

II. Problem Description

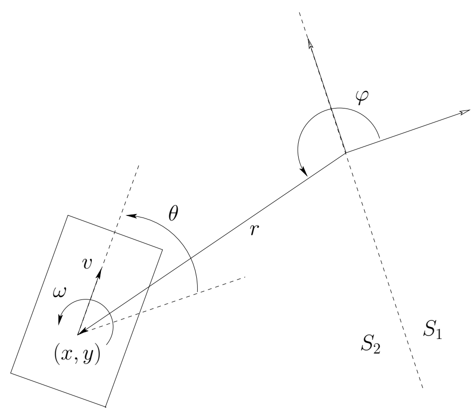

Fig. 1. A mobile robot tracking a moving target.

Consider a mobile robot like the one depicted in Figure 1. The robot is supposed to maintain a constant position and orientation with respect to another moving target which can be thought of as its leader. The configuration of the robot can be specified with respect to its tracking point by its position difference, (x, y), and the orientation difference, θ. Assume that the vehicle should move with translational velocity v_d and rotational velocity ω_d in order to minimize these differences, i.e. stabilize the variables (x, y, θ) to the origin. These speeds are to be realized through the translational and rotational acceleration inputs, a and α, respectively. Let e_v and e_ω denote the velocity errors, between the true vehicle velocities and the desired. Then the dynamics of a mobile robot can simply be described as follows:

$$\dot{x} = (v_d + e_v)\cos\theta \tag{1a}$$
$$\dot{y} = (v_d + e_v)\sin\theta \tag{1b}$$
$$\dot{\theta} = \omega_d + e_\omega \tag{1c}$$
$$\dot{e}_v = a \tag{1d}$$
$$\dot{e}_\omega = \alpha. \tag{1e}$$

Using polar coordinates for the description of the position and orientation:

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan(y/x), \quad e_\theta = \theta - 2\varphi$$

the dynamics of the robot can be written as:

$$\dot{r} = (v_d + e_v)\cos(e_\theta + \varphi) \tag{2a}$$
$$\dot{\varphi} = \frac{v_d + e_v}{r}\sin(e_\theta + \varphi) \tag{2b}$$
$$\dot{e}_\theta = \omega_d + e_\omega - 2(v + e_v)\sin(e_\theta + \varphi) \tag{2c}$$
$$\dot{e}_v = a \tag{2d}$$
$$\dot{e}_\omega = \alpha. \tag{2e}$$

Let us consider the regions, $S_1 = \{(r, \varphi, \theta, v, \omega) \mid \cos\varphi > 0\}$ and $S_2 = \{(r, \varphi, \theta, v, \omega) \mid \cos\varphi < 0\}$, and restrict our analysis to the domain $S_1 \cup S_2$. Note, that by the definition of φ, the origin belongs to S_1. The reason for excluding the two y semi-axes is to ensure arbitrarily large but bounded control inputs. These, in turn can establish positive invariance for each one of the sets S_1 and S_2. When only bounded inputs are allowed, the analysis has to be restricted to $S_1^\delta = \{(r, \varphi, \theta, v, \omega) \mid \cos\varphi > \delta\}$ and $S_2^\delta = \{(r, \varphi, \theta, v, \omega) \mid \cos\varphi < -\delta\}$, for a sufficient $\delta > 0$.

Due to the invariance of S_1 and S_2, it makes sense to define the following metric on S_1 and S_2:

Proposition II.1 *The function $d_1 : S_i \to \mathbb{R}$, $i = 1, 2$ given as:*

$$d_c(r, \varphi, \theta, e_v, e_\omega) \triangleq \sqrt{\frac{r^2}{\cos^2\varphi} + \sin^2\varphi + \theta^2 + e_v^2 + e_\omega^2} \tag{3}$$

defines a metric on S_i.

Proof: In order for (3) to qualify for a metric on S_i, it has to satisfy: (i) $d_c(z) \geq 0, \forall z \in S_i$, (ii) $d_c(z) = 0 \Leftrightarrow z = 0, \forall z \in S_i$, and (iii) $d_c(z_1+z_2) \leq d_c(z_1)+d_c(z_2), \forall z_1, z_2 \in S_i$. We will show that this is the case for S_1; the case for S_2 follows similarly. Properties (i)-(ii) are obvious. For (iii) we only to consider any two vectors in S_1, z_1 and z_2, substitute in Cartesian coordinates and take the difference $d_c(z_1+z_2)-d_c(z_1)-d_c(z_2)$. In S_1, for this difference to be negative, it suffices to show that $\frac{r}{\cos\varphi} - \frac{r_1}{\cos\varphi_1} - \frac{r_2}{\cos\varphi_2} < 0$. For the latter:

$$\frac{r}{\cos\varphi} - \frac{r_1}{\cos\varphi_1} - \frac{r_2}{\cos\varphi_2} = -\frac{(x_2 y_1 - x_1 y_2)^2}{x_1 x_2 (x_1 + x_2)}$$

which is negative in S_1 since $x_1, x_2 > 0$. Similarly it is shown for S_2. ■

Figure 6 shows the topology induced by the metric on the (x, y) plane. Note that this metric

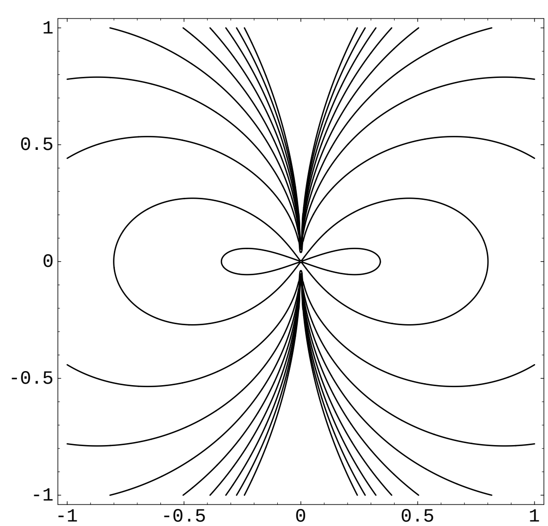

Fig. 2. The (x, y)-topology induced by the metric.

applies in S_1 and S_2, but not in $S_1 \cup S_2$. Our stability results will be expressed with respect to this metric.

III. Closed Loop Stability

A. The Singularly Perturbed System

By an appropriate choice of control inputs and sufficiently large gains, the system (2) can take the form of a singularly perturbed system. Let the desired velocities be defined as:

$$v_d = -\frac{k_1 r}{\cos\varphi} \tag{4a}$$

$$\omega_d = 2(v_d + e_v)\sin(e_\theta + \varphi) - k_\omega e_\theta, \tag{4b}$$

and the acceleration inputs as:

$$a = -k_2 e - \frac{r^2 \sec^3\varphi + \cos\varphi\ \sin^2\varphi}{r} \tag{5a}$$

$$\alpha = -e_\theta - k_\alpha e_\omega, \tag{5b}$$

If we let $k_\alpha = k_\omega k_\theta$, with $k_\theta > 1$ and substitute (4b) and (5b) into (2) we obtain a singular perturbed version of (2), the (e_θ, e_ω) subsystem of which is:

$$\frac{1}{k_\omega}\begin{bmatrix}\dot{e}_\theta \\ \dot{e}_\omega\end{bmatrix} = -\begin{bmatrix} 1 & -\frac{1}{k_\omega} \\ \frac{1}{k_\omega} & k_\theta \end{bmatrix}\begin{bmatrix} e_\theta \\ e_\omega \end{bmatrix},$$

giving rise to an exponentially stable boundary layer system:

$$\frac{\mathrm{d}e_\theta}{\mathrm{d}\tau} = -e_\theta, \qquad \frac{\mathrm{d}e_\omega}{\mathrm{d}\tau} = -k_\theta e_\omega \tag{6}$$

where $\tau = t/\epsilon$ and $\epsilon \triangleq \frac{1}{k_\omega}$ is the singular parameter, and the reduced system

$$\dot{r} = (v_d + e_v)\cos\varphi \tag{7a}$$

$$\dot{\varphi} = \frac{v_d + e_v}{r}\sin\varphi \tag{7b}$$

$$\dot{e}_v = a. \tag{7c}$$

Applying (4a) and (5a) in (7), the closed loop reduced system becomes:

$$\dot{r} = e_v\cos\varphi - k_1 r \tag{8a}$$

$$\dot{\varphi} = \frac{e_v \sin\varphi}{r} - k_1\tan\varphi \tag{8b}$$

$$\dot{e}_v = -k_2 e_v - \frac{r^2\sec^3\varphi + \cos\varphi\ \sin^2\varphi}{r}, \tag{8c}$$

which is easily shown to be exponentially stable with respect to the metric (3) after considering the Lyapunov function candidate:

$$V_r(r, \varphi, e_v) \triangleq \frac{r^2}{\cos^2\varphi} + \sin^2\varphi + e_v^2 \tag{9}$$

and taking its time derivative:

$$\dot{V}_r = -2k_2e_v^2 - 2k_1\left(\frac{r^2\sec^2\varphi}{\cos^2\varphi} + \sin^2\varphi\right) \leq -2\min\{k_1,k_2\}d_c^2(r,\varphi,e_v).$$

From the exponential stability of (6) and (7) we can conclude that there is a sufficiently large k_ω for which (2) is exponentially stable with respect to the metric (3). Estimating the bound for k_ω is more involved and is treated in the next Section.

B. Gain Selection for Stability

The stability analysis will be performed here assuming that the system is within the region where $\cos\varphi > 0$. The case where $\cos\varphi < 0$ can be treated similarly. In the course of the discussion it will become clear that each one of these regions are made positively invariant through (5a)-(5b).

A Lyapunov function for the boundary layer system (6) can be the following:

$$V_b(e_\theta,e_\omega) \triangleq \tfrac{1}{2}e_\theta^2 + \tfrac{1}{2k_\theta}e_\omega^2 \tag{10}$$

Combining (10) with (9) we can define a Lyapunov function for the singular perturbed system:

$$V = r^2\sec^2\varphi + \sin^2\varphi + e_v^2 + \tfrac{1}{2}(e_\theta^2 + \tfrac{e_\omega^2}{k_\theta}) \tag{11}$$

Its time derivative, $\dot{V}$ is:

$$\begin{aligned}\dot{V} = &-\frac{k_\theta-1}{2k_\theta}(e_\theta-e_\omega)^2 - \frac{1+2k_\alpha-k_\theta}{2k_\theta}(e_\theta^2+e_\omega^2)\\ &+\frac{e_v\sin(2\varphi)[\sin(\varphi+e_\theta)-\sin\varphi]}{r}\\ &+2k_1\cos e_\theta\ \sin^2\varphi - k_1\sin(2\varphi)\sin e_\theta - 2k_2e_v^2\\ &-2e_vr\sec^3\varphi\ \sin^2(\tfrac{e_\theta}{2}) - 2k_1r^2\cos e_\theta\ \sec^4\varphi.\end{aligned}$$

Using the fact that $|e_\theta| \leq \frac{\pi}{2}$, implying $\cos e_\theta \geq 0$ and $|e_\theta| \geq |\sin e_\theta|$, we can bound $\dot{V}$ as follows:

$$\begin{aligned}\dot{V} \leq &-\frac{k_\theta-1}{2k_\theta}(e_\theta-e_\omega)^2 - \frac{1+2k_\alpha-k_\theta}{2k_\theta}(e_\theta^2+e_\omega^2)\\ &-2\min\{k_1,k_2\}\cos e_\theta(e_v^2+r^2\sec^4\varphi+\sin^2\varphi)\\ &+2\max\{1,k_1\}|e_\theta|\cos\varphi(1+\tfrac{|e_v|}{r})(|\sin\varphi|+r^2\sec^4\varphi).\end{aligned}$$

From the local Lipschitz continuity $g(x) = x^2$ it follows that there exists a positive constant L such that:

$$|r^2\sec^4\varphi| \leq L|r\sec^2\varphi| = Lr\sec^2\varphi$$

from which we can extend the bound of $\dot{V}$:

$$\begin{aligned}\dot{V} \leq &-\frac{k_\theta-1}{2k_\theta}(e_\theta-e_\omega)^2 - \frac{1+2k_\alpha-k_\theta}{2k_\theta}(e_\theta^2+e_\omega^2)\\ &-2\min\{k_1,k_2\}\cos e_\theta(e_v^2+r^2\sec^4\varphi+\sin^2\varphi)\\ &+2(1+L)\max\{1,k_1\}|e_\theta|\cos\varphi(1+\tfrac{|e_v|}{r})\\ &(|\sin\varphi|+r\sec^2\varphi+|e_v|).\end{aligned}$$

A necessary step in order to bound $\dot{V}$ further is taken with the following Lemma:

Lemma III.1 *The term $\frac{e_v}{r}$ is upper bounded by a positive constant c.*

Proof: Boundedness of $\frac{e_v}{r}$ follows from the stability of (8). If $frac1k_2$ is treated as a singular parameter then (8) is transformed to a boundary layer system $\frac{de_v}{d\tau} = -e_v$ and a reduced system:

$$\dot{r} = -rk_1 \qquad \dot{\varphi} = -k_1\tan\varphi,$$

which is also exponentially stable. Therefore, for a sufficiently large k_2, the origin of (8) is exponentially stable. The time scale decomposition of (8) imposed by the increase in k_1 ensures that $|e_v|$ reduces much faster than r, making the ratio $\frac{|e_v|}{r}$ converge to zero exponentially. This exponential convergence implies that there are positive constants, c and m such that

$$\frac{|e_v|}{r} \leq ce^{-mt} \Rightarrow \frac{|e_v|}{r} \leq c, \quad \forall t \geq 0$$

■

Using the bound on $\frac{|e_v|}{r}$ suggested by the aforementioned Lemma, we can obtain the following bound for $\dot{V}$:

$$\dot{V} \leq -\lambda_{\min}(Q)d_c^2(r,\varphi,e_v,e_\theta,e_\omega) \tag{12}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue and the symmetric matrix Q is given as:

$$Q \triangleq \begin{bmatrix} \frac{1-2k_\alpha-k_\theta}{2k_\theta} & \sqrt{2}(1+L)\max\{1,k_1\}(1+c) \\ \sqrt{2}(1+L)\max\{1,k_1\} & 2\min\{k_1,k_2\}\cos\left(e_\theta(0)\right)\end{bmatrix}$$

We then need to make sure that Q is positive definite. First note that without loss of generality, one can assume that $k_2 > k_1$, which is reasonable since we need k_2 to be sufficiently large. Then in order for Q to have positive eigenvalues, it is sufficient that

$$k_\alpha > \frac{\max\{1,k_1\}^2 \sec\big(e_\theta(0)\big)\ k_\theta(1+c)}{k_1} + \frac{1}{2}(k_\theta - 1)$$

Then, the smallest eigenvalue is:

$$\begin{aligned}\lambda_{\min}(Q) = &-\frac{1}{4} + \frac{k_\alpha}{2k_\theta} + (1+L)k_1\cos\big(e_\theta(0)\big)\\ &+ \Big[16k_\theta[2(1+c)^2\max\{1,k_1\}^2 k_\theta\\ &- (1+L)\cos\big(e_\theta(0)\big)k_1(1+2k_\alpha - k_\theta)]\\ &+ [1+2k_\alpha + \big(4(1+L)k_1\cos(e_\theta(0)) - 1\big)k_\theta]^2\Big]^{1/2}\end{aligned}$$

Therefore, with respect to metric d_c, V reduces exponentially. Using the Comparison Lemma, we conclude for $\lambda \equiv \lambda_{\min}(Q)$:

$$\begin{aligned}&d_c\big(r(t),\varphi(t),e_v(t),e_\theta(t),e_\omega(t)\big)\\ &\leq 2k_\theta V\big(r(t),\varphi(t),e_v(t),e_\theta(t),e_\omega(t)\big) \leq 2k_\theta V(0)e^{-\lambda t}\\ &\leq 2k_\theta d_c\big(r(0),\varphi(0),e_v(0),e_\theta(0),e_\omega(0)\big)e^{-\lambda t}\end{aligned}$$

IV. Acceleration Perturbations

This section will demonstrate that under acceleration input disturbances, the state of the closed loop system (1)-(4)-(5) is ultimately bounded with respect to the metric (3), by a $\mathcal{K}$ class function of the magnitude of the disturbances. This implies that the system is input-to-state stable with respect to this metric. Such a property is quite important in view of recent results that link input-to-state stability to vehicle formations, and allow the quantitative characterization of the performance and robustness properties of a formation.

Let the closed loop system (1)-(5) be perturbed by acceleration disturbances $\delta \triangleq (\delta_a, \delta_\alpha)^T$:

$$\dot r = (v+e_v)\cos(e_\theta+\varphi) \tag{13a}$$
$$\dot\varphi = \frac{v+e_v}{r}\sin(e_\theta+\varphi) \tag{13b}$$
$$\dot e_v = a + \delta_a \tag{13c}$$
$$\dot e_\theta = -k_\omega e_\theta + e_\omega \tag{13d}$$
$$\dot e_\omega = -e_\theta - k_\alpha + \delta_\alpha \tag{13e}$$

Then, the time derivative of the Lyapunov function (11) will have two new terms:

$$\begin{aligned}\dot V = &-\frac{k_\theta - 1}{2k_\theta}(e_\theta - e_\omega)^2 - \frac{1+2k_\alpha - k_\theta}{2k_\theta}(e_\theta^2 + e_\omega^2)\\ &+ \frac{e_v\sin(2\varphi)[\sin(\varphi+e_\theta) - \sin\varphi]}{r}\\ &+ 2k_1\cos e_\theta\ \sin^2\varphi - k_1\sin(2\varphi)\sin e_\theta - 2k_2 e_v^2\\ &- 2e_v r\sec^3\varphi\ \sin^2(\tfrac{e_\theta}{2}) - 2k_1 r^2\cos e_\theta\ \sec^4\varphi\\ &+ 2e_v\delta_a + \frac{e_\omega\delta_\alpha}{k_\theta}.\end{aligned}$$

and can similarly be bounded from above by:

$$\begin{aligned}\dot V &\leq -\lambda d_c^2 + 2e_v\delta_a + \frac{e_\omega\delta_\alpha}{k_\theta}\\ &\leq -\lambda d_c^2 + \sqrt{2}\max\{2,\tfrac{1}{k_\theta}\}d_c\,\|\delta\|_2\end{aligned}$$

For a constant parameter $\zeta \in (0,1)$, we can then have:

$$\dot V \leq -\lambda(1-\zeta)d_c^2 + (\sqrt{2}\max\{2,\tfrac{1}{k_\theta}\}\,\|\delta\|_2 - \lambda\zeta d_c)d_c$$

which is negative, provided that

$$d_c \geq \frac{\sqrt{2}\max\{2,\frac{1}{k_\theta}\}}{\lambda\zeta}\,\|\delta\|_2$$

Treating (13) as a perturbed system, we have that:

$$d_c \leq d_c(0)e^{-\lambda t} + \frac{\sqrt{2}\max\{2,k_\theta^{-1}\}}{\lambda\zeta}\,\|\delta\|_2$$

which implies that (13) is ISS with respect to the input δ and the norm induced by the metric d_c.

V. Numerical Validation

In this section we verify the ISS properties of the dynamic model (1) with respect to the metric chosen. In the simulation scenario, the initial conditions for the system are set to $(x, y, \theta, v, \omega) = (0.1, 0.1, -\frac{\pi}{2}, 0, 0)$ (Figure 3). The controller gains were set to $k_1 = 1$, $k_\omega = 5$, $k_2 = 100k_1$, and $k_\theta - 2$. The system is perturbed by sinusoidal acceleration disturbances of the form $\delta_a = 50\sin(100t)$ and $\delta_\alpha = 50\cos(100t)$. Under these initial conditions and disturbances, the control scheme proves to be robust, ensuring convergence of the system state to the origin. The acceleration disturbances force the robot to chatter along its path, as shown in Figure 3, however stability is maintained.

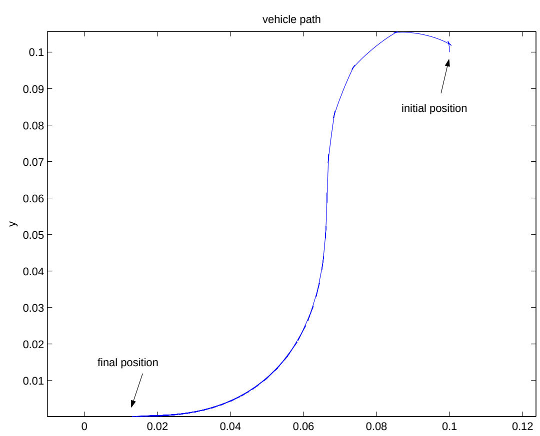

Fig. 3. Path of the mobile robot.

Figures 4 and 5 show more explicitly the effect of these disturbances on the position and velocity, errors of the robot respectively. Figure 6 shows the evolution of the metric d_c along the trajectory of the robot, indicating clearly the initial transient phase where it the metric is decreasing and then the steady state, where it is ultimately bounded.

VI. Conclusion

In this paper we establish the ISS properties of dynamic unicycle models for mobile robots, with respect to acceleration input disturbances. The system is rendered ISS after the application

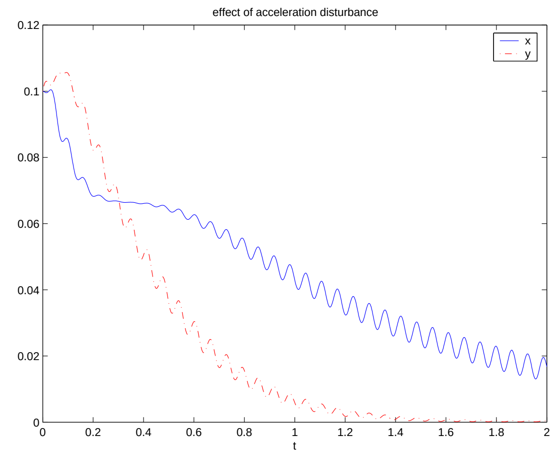

Fig. 4. The effect of acceleration disturbances on position.

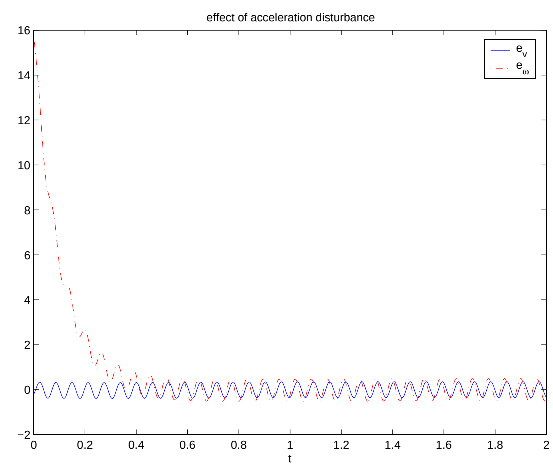

Fig. 5. The effect of acceleration disturbances on velocity errors.

of a (discontinuous) feedback controller that is shown to be robust with respect to such perturbations. The ISS properties of the closed loop system are established with respect to a particular metric, the induced topology of which seems suits the nonholonomic nature of the system. We believe that this result is conceptually important because it shows that under certain conditions and in an appropriate topology, systems that may not be feedback linearizable may still enjoy ISS properties. This work could find application in formation control of mobile robots, in view of recent developments in this field.

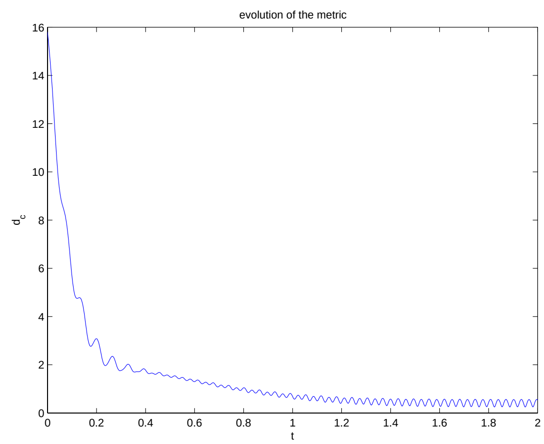

Fig. 6. The evolution of the metric along the trajectory.

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