Multi-agent navigation functions revisited

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***Abstract*— This paper presents a methodology for designing a control law that can provably steer a group of mobile agents to a free-floating formation of specific shape, while avoiding collisions. We indicate why existing multi-agent solutions have a theoretical limitation due to one of the working assumptions of the single-agent potential function approach being violated. A new nonsmooth type of multi-agent potential functions is thus developed. It is shown how this extension of the navigation function concept to multi-agent systems ensures the non-degeneracy of the critical points of the potential function.**

I. Introduction

With the advent of inexpensive and fast computation and communication electronics that enable the distributed application of formation control algorithms in multi-robot systems, the interest in the theoretical foundations of control design for formation stabilization intensified. Several design alternatives have appeared in literature, and a partial list can include [1]–[5]. The focus of the paper is on a specific methodology that generalizes a single robot potential field approach to groups of robots. This methodology promises algorithmic completeness, combined motion planning and feedback control, and global convergence guarantees. It is based on an artificial potential function, known as a *navigation function* [6], which can be tuned so that it does not have local minima. Navigation functions for single robots are introduced in [7] for environments with sphere-world topology, and are generalized to star-world environments through a series of diffeomorphic transformations in [6]. The coverage of related literature in this paper is limited to alternative *multi-agent* generalizations of this navigation function approach.

Such generalizations have appeared in literature both in the form of centralized and decentralized schemes. A centralized architecture [8]–[11] typically involves a single potential function that depends on the configuration of all formation members. The gradient of this function is either computed at a central location and then communicated to the group members, or computed independently by each individual member with knowledge of the full system state. The approach in [8] is one of the earliest reported, it applies to holonomic sphere-like robots, but does not offer a proof that the potential function is a navigation function. References [10], [11] prove some of the navigation function properties of the potential conjectured in [8], and address issues related to nonholonomic constraints; [10] does so at a kinematic level using a nonsmooth potential function construction, while [11] extends this technique to dynamical models. A practical limitation of a centralized approach is that due to the dependence on full state information, scaling to large groups can be problematic.

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To meet this challenge, decentralized architectures require state information from a subset of the robots in the group, typically the nearest neighbors and the robots relative to which an individual robot needs to position itself. In this category [12]–[17], individual agents use their own potential function, calculated using locally available information (either sensed or communicated by neighbors). The challenge here is in proving that minimizing all these (local) potential functions leads the system to a single desired equilibrium. The constructions in [12], [17] and [13] express the possible collision configurations by a product of relation functions—similarly to [7], [8] where functions quantify proximity to each potential collision.

In [14] the workspace includes agents as well as obstacles, with each agent having a limited communication range. In [15] and [16], the problem of formation control is treated in a scenario where agent relative position specifications are expressed through a directed graph. In [15], decentralization is limited because each agent has a copy of a centralized potential function. In [16], the degree of decentralization is increased since the potential function is decomposed into functions dependent on locally available information. In both [15] and [16], the communication range of each agent is assumed limited.

All the above generalizations of the single-robot navigation function make the same assumptions and follow the analysis steps of the original construction in [7] as suggested first in [8] and later refined in [9], [10], [17]. It turns out, however, that some of these assumptions unavoidably break down in the multi-agent case. One of these assumptions is the one that requires that "obstacles are isolated." In the multi-robot case, collisions can occur between any combination of robots and since all robots can move, it is not clear why only two robots can collide with each other at any given time. Numerical results so far have not contradicted such a claim, but the underlying theory supporting existing generalizations cannot guaranteed it.

This paper suggests an alternative construction. First, agents' proximity to collision is not quantified using a *product* of pairwise proximity functions. Second, formations "float" in space (achieve translational invariance) avoiding mixing absolute and relative position coordinate systems. Third, the collision proximity function is nondifferentiable. In addition, it is shown how both smooth and nonsmooth critical points can be made non-degenerate by an appropriate assignment of the potential functions' parameters. Finally, a new (nonsmooth) formation controller is proposed and its convergence properties shown using stability results for differential inclusions.

II. Single Agent Navigation Functions

In its original form, the *navigation function* [6] was defined for one robot agent with single integrator dynamics of the form $\dot{x} = u$, moving in a workspace $\mathcal{W} \triangleq \{x \in \mathbb{R}^n : \|x\| \leq \rho_0\}$, populated by M spherical obstacles $\mathcal{O}_i \triangleq \{x \in \mathbb{R}^n : \|x - c_i\| < \rho_i\}$. Here, c_i and ρ_i are the center and radius of obstacle i, respectively, and ρ_0 denotes workspace $\mathcal{W}$ outer boundary. The free workspace where the robot can move is the set $\mathcal{F} \triangleq \mathcal{W} \setminus \bigcup_{i=1}^{M} \mathcal{O}_i$. We use the notation $\delta\mathcal{F}$ to denote the boundary of a set—in this case, $\mathcal{F}$—reserving the symbol ∂ for (generalized) gradients.

Workspace $\mathcal{F}$ is said to be *valid* if the closures of all $\mathcal{O}_i$ are in the interior of $\mathcal{F}$, and that none of them intersect, that is, $\|c_i - c_j\| > \rho_i + \rho_j$.

Definition 1 ([6]): Let $\mathcal{F} \subset \mathbb{R}^n$ be a compact connected analytic manifold with boundary. A map $\varphi : \mathcal{F} \to [0, 1]$ is a navigation function if φ is

1) analytic on $\mathcal{F}$;
2) polar on $\mathcal{F}$, with minimum at x_d;
3) Critical points in $\mathcal{F}$ are non-degenerate[1];
4) admissible on $\mathcal{F}$ (uniformly maximal on $\delta\mathcal{F}$).

Rimon and Koditschek [7] have shown that if the workspace is valid, parameter κ can be selected so that the following function can become a navigation function

$$\varphi = \frac{\gamma_d}{({\gamma_d}^{\kappa} + \beta)^{1/\kappa}} \quad . \tag{1}$$

This function is essentially a "squashed" version of $\frac{\gamma_d}{\beta}$, bounded to vary in the $[0, 1]$ interval, and passed through a filter of the form $y = x^{1/\kappa}$ in order to fix the otherwise degenerate nature of x_d. In the above, $\gamma_d(x)$ is a scalar function that serves as a metric of how close the robot is to its desired configuration x_d, $\beta(x)$ is another scalar function that quantifies how close the robot is to obstacles, and κ is a tuning parameter, which needs to be sufficiently large in order for the function to have the properties of Definition 1. In addition, γ_d and β are assumed to have a special structure, namely

$$\gamma_d(x) \triangleq \|x - x_d\|^2, \qquad \beta_i(x) \triangleq \|x - c_i\|^2 - \rho_i^2, \tag{2a}$$

$$\beta(x) \triangleq \prod_{i=1}^{M} \beta_i(x) \quad , \tag{2b}$$

where i ranges from 1 to M—the zero-index obstacle is included in the workspace description here.

In the particular case of φ given by (1) and with the assumption that the workspace is valid, one can analytically show that φ is a navigation function [7]. The workspace $\mathcal{F}$ is first decomposed into regions (ϵ is thought of as a small parameter):

- $\{x_d\}$: the destination point,
- $\mathcal{B}_i(\epsilon)$: Region "close" to the boundary of obstacle i, where $0 < \beta_i < \epsilon$,
- $\delta\mathcal{F}$: Boundary of free space,
- $\mathcal{F}_0(\epsilon) \triangleq \bigcup_{i=1}^{M} \mathcal{B}_i \setminus \{x_d\}$: Region "close" to the obstacle boundary,
- $\mathcal{F}_2(\epsilon) \triangleq \mathcal{F} \setminus (\{x_d\} \cup \delta\mathcal{F} \cup \mathcal{F}_0)$: away from obstacles,

and then a sequence of propositions establish that

1) the destination x_d is a non-degenerate minimum of φ,
2) the critical points of φ are in the interior of the free space,
3) for every ϵ one can choose a κ so that $\frac{\gamma}{\beta}$ has no critical points away from obstacles,
4) there exists a lower bound for ϵ, below which the critical points of $\frac{\gamma}{\beta}$ close to the obstacle boundary are not local minima,
5) there is another lower bound for ϵ, below which there are no critical points close to the workspace boundary, and that
6) there is an ultimate lower bound for ϵ, below which whatever critical points other than the destination are inside the free space, are non-degenerate (and therefore, they have to be saddles).

In summary, there is an upper bound on the value of ϵ, associated with an lower bound on κ, for which φ is a navigation function on $\mathcal{F}$, provided that the workspace is valid.

[1] Critical points of a differentiable function are non-degenerate if the Hessian matrix is nonsingular there. In the case of nondifferentiable functions, the notion of non-degeneracy for critical points is discussed in Section IV.

III. Multi-Robot Navigation Functions

A. The initial approach

Consider now N agents. The notation is changed slightly from Section II to denote this fact, and for simplicity we assume that there are no stationary obstacles in the agents' environment. Rather, the agents may run into each other, and therefore each one of them is an obstacle to all others. The dynamics of each agent have, as before, the form of single integrators

$$\dot{p}_i = u_i, \quad p_i \in \mathbb{R}^n, \quad i = 1, \ldots, N \tag{3}$$

where p_i and u_i are the position and control input of agent i, respectively. Let p denote the stack vector of all p_i, and define $\mathcal{P}$ as the set where $\|p_i - p_j\| \geq \rho$, for all $i, j = 1, \ldots, N$. Here, $\mathcal{P}$ is in the role of $\mathcal{F}$.

An approach followed in existing literature is to start with (1), define a goal function γ_d and the obstacle function β in a similar way as in (2) and then set

$$\varphi = \frac{\gamma}{(\gamma^{\kappa} + \beta)^{1/\kappa}} \quad .$$

These choices have advantages and disadvantages: on one hand, a γ_d defined as in (2) requires each robot to achieve a pre-specified position in the workspace and thus does not allow the formation to "float" freely in space; on the other hand it ensures that the destination point will eventually turn out to be a non-degenerate critical point in $\mathcal{P}$. The choice of β, however, as the product of individual obstacle functions that vanish at a collision configuration, has more subtle implications: it affects the validity of workspace $\mathcal{P}$, because now the obstacles are *not* isolated. In fact, there is always the case where a sufficient number of individual obstacle functions vanish concurrently, forcing $\nabla^2\beta$ to vanish too, which prevents one from proving claim 4) in the list of the previous section, at least using the known approach. This is because at a critical point,

$$\nabla^2 \left(\frac{\gamma}{\beta}\right) \propto \left(2\frac{\|\nabla\beta\|}{\|\nabla\gamma\|}\right) I - \nabla^2\beta \quad ,$$

and with $\nabla^2\beta \to 0$, and I being the identity matrix, the hope of finding a negative eigenvalue for the Hessian diminishes.

To see this through an example, take the case of a single dimensional obstacle function product $\beta(x)$, made up of four factors: $\beta_0(x), \ldots, \beta_3(x)$. Express the second derivative of $\beta(x) = \beta_0(x) \cdot \beta_1(x) \cdot \beta_2(x) \cdot \beta_3(x)$, and verify that when $\beta_0 = \beta_1 = \beta_2 = 0$, it is $\beta''(x) \equiv 0$, irrespectively of β_3.

Having said this, it is not necessarily the case that a potential function built in this way may fail to stabilize a multi-agent formation. However unlikely as it might be for any critical points to creep into a region where a sufficient number of β_{ij} vanish simultaneously (such as the origin of the space of relative positions) as the tuning parameter κ increases, this possibility cannot be excluded. Without the guarantees about the nature of critical points that the original navigation function approach offers, the question as to whether any trajectory along the negated gradient converges to the desired configuration, is open.

B. The new suggestion

Suppose that the robots in a formation can be modeled as autonomous, identical sphere-shaped agents of diameter d_0, having kinematics (3). Assume that the only possible collisions that can occur are between them. Relax the requirement for having analytic and admissible functions; it appears that there can be benefits in using nonsmooth functions to construct φ, and we intend to exploit them.

The goal is to steer the agents from any relative initial configuration into a pre-specified formation, while avoiding inter-agent collisions, but *without fixing the location* of that formation in space. The desired formation is described by means of a graph:

Definition 2 (Formation graph [16]): The formation graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{C}\}$ is a directed labeled graph consisting of

- a set of vertices $\mathcal{V} = \{v_1, \ldots, v_N\}$, indexed by the mobile agents,
- a set of edges $\mathcal{E} = \{(i,j) \in \{1, \ldots, N\} \times \{1, \ldots, N\}\}$, containing ordered pairs of nodes that represent inter-agent position specifications, and
- a set of labels (formation specifications) $\mathcal{C} = \{c_{ij} \mid (i,j) \in \mathcal{E}\}$.

Whenever $c_{ij} \in \mathcal{C}$, its existence implies that the desired relative *position* between agent i and agent j is c_{ij}, that is, we should ideally have $\|p_i - p_j - c_{ij}\| = 0$. If graph $\mathcal{G}$ is (weakly) connected, then the formation is uniquely specified.

Meeting the formation specifications depends on relative positions only, and our analysis is therefore performed in the space of relative differences $q_{ij} \triangleq p_i - p_j$. If we denote q the stack vector of relative agent positions, then $q = (B^\mathsf{T} \otimes I)p$, where B is the incidence matrix of graph $\mathcal{G}$, and $\otimes$ denotes the Kronecker matrix product. Since the maximum number of possible edges in a graph of size N is $\frac{N(N-1)}{2}$, the workspace in this case is

$$\mathcal{Q} = \mathbb{R}^{nN(N-1)/2} - \{q \mid \|q\| > \rho_0 \text{ and } \|q_{ij}\| \leq d_0,\ \forall i,j \in \{1,\ldots,N\}\} \ ,$$

that is, it excludes collision configurations between agents and configurations where agents are too far apart from each other to be regarded as a group. We assume a decomposition of $\mathcal{Q}$, similar to that of $\mathcal{F}$ shown in Section III-A:

- $\mathcal{B}_q(\epsilon) \triangleq \{q \in \mathcal{Q} \mid 0 < \beta(q) < \epsilon\}$
- $\mathcal{Q}_0 \triangleq \mathcal{B}_q(\epsilon)$
- $\mathcal{Q}_2 \triangleq \mathcal{Q} \setminus \big(\delta\mathcal{Q} \cup \mathcal{B}_q(\epsilon)\big)$.

The function φ which we use as potential is not a navigation function in the strict sense since it lacks some of the properties listed in Section II, but we show that it is still polar, its critical points are non-degenerate, and it has the structure of the "seed" function used in [7] to generate a navigation function:

$$\bar{\varphi}(q) = \varphi(q)^{\frac{1}{\kappa}} \ , \tag{4}$$

where

$$\varphi(q) = \frac{\gamma(q)}{\beta(q)} \ , \tag{5}$$

in which γ is the "goal" function, a positive definite (in $\mathcal{Q}$) scalar function attaining zero only when the agents are at their desired relative configurations

$$\gamma(q) = {\gamma_d}^{\kappa}(q), \quad \text{with} \quad \gamma_d(q) \triangleq \|q - c\|^2 \ , \tag{6}$$

and c is the constant stack vector of formation specifications. Function $\beta(q)$ is a positive semidefinite scalar function, varying in the interval $[0, 1]$, vanishing when any agents are in contact and achieving its maximum when all agents are *sufficiently* far from each other. We suggest the following choice of the scalar function β:

$$\beta(d) = \log\left(\mu - a\, e^{-\left(-r + d + d^2\right)^2}\right) \ , \tag{7}$$

with $\mu > 2$, and a, r positive scalar parameters that are set to fix the location where β vanishes, and determine its derivative there. For example, if one needs β to vanish when $d = d_0$ and have a derivative there equal to ζ, then the choice

$$r = d_0^2 + d_0 + \frac{\zeta}{2(1-\mu)(1+2d_0)} \tag{8a}$$

$$a = (\mu - 1)\, e^{\left([{d_0}^2 + d_0 - r]^2\right)} \ , \tag{8b}$$

achieves this goal. The argument in β is the minimum out of all pairwise distances between agents

$$d(q) \triangleq \min_{\substack{i \in \{1,\ldots,N\} \\ j \neq i}} \{\|q_{ij}\|\} \ , \tag{9}$$

measured between the centers of their spherical shapes, and where the minimum is taken over every combination of $i, j \in \{1, \ldots, N\}$. The properties of this distance function are important in the present analysis and will be discussed in some detail in Section III-C.

The choice of β as in (7) gives the obstacle function the following attributes:

1) it vanishes whenever any two agents collide and remains positive otherwise;
2) it approaches a constant asymptotically as the distance between agents increases;
3) it is nondifferentiable with respect to q, due to $d(q)$.

The nonsmooth character of β is due to the dependence on the distance function $d(q)$, which is inherently nondifferentiable. The use of the (nonsmooth) distance function here is motivated by the desire to avoid the multiplication of pair-wise agent collision metric functions.

In view of (9) and (5), the proposed potential function can be expressed as a pointwise maximum function

$$\varphi(q) = \max_{\substack{i\in\{1,\ldots,N\}\\ j\neq i}} \left\{ \frac{\gamma(q)}{\beta(\|q_{ij}\|)} \right\} \ ,$$

among a set in which each function is differentiable everywhere in the interior of $\mathcal{Q}$.

The formation specification c will be called *valid* if

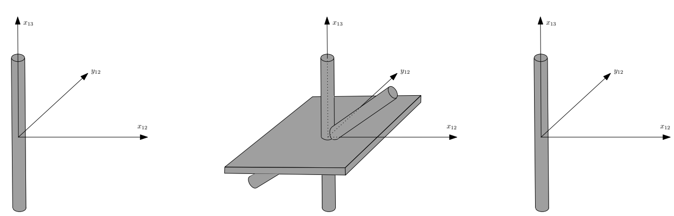
$$\min_{\mathcal{Q}_0} \|\nabla \gamma_d\| > \frac{\max_{\mathcal{Q}_0} \|\nabla \beta\|}{\min_{\mathcal{Q}_0} \left\{ \frac{\partial \beta}{\partial d} \right\}} \ .$$

Intuitively, this requires that desired formation is sufficiently away from collision configurations.

C. The distance function

The distance function d of a point x to a set Ω is defined as the minimum norm of the difference between the point and any other point in the set: $d(x) \triangleq \min_{z\in\Omega} \|x - z\|$. Comparing to (9), x is identified with q and the set Ω is the manifold where any of the components of q becomes zero.

With the exception of trivial cases, q cannot be continuously visualized in three dimensions. The easiest case that can be reasonably depicted is that of three planar agents. The formation configuration can be uniquely described in terms of the horizontal and vertical relative position of agents 1 and 2, x_{12} and y_{12}, respectively, and the horizontal and vertical position of agents 1 and 3, x_{13} and y_{13}, respectively. In this case, q is (still) four-dimensional. But we can attempt to visualize the d_0-level sets of (9) using three-dimensional slices of this four-dimensional space (see Fig. 1).



(a) $y_1 - y_3 < 0$ (b) $y_1 - y_3 = 0$ (c) $y_1 - y_3 > 0$

Fig. 1. Three 3-D slices of an inter-agent distance function defined in a four-dimensional space. The collision configurations between agents 1 and 2 are marked by the cylinder that contains the (hyper)line $x_{12} = y_{12} = 0$. In the three-dimensional slices where the dimension y_{13} is not pictured, the collision configurations between agents 1 and 3 are shown as the "thick" hyperplane passing through the origin on the $y_1 - y_3 = 0$ slice. Note the diagonal cylinder with axis on the x_{13}-x_{12} plane: this represents collisions between agents 2 and 3 (although x_{23} and y_{23} are not mapped). This diagonal collision region expresses the fact that when $q_{12} = q_{13}$, agents 2 and 3 overlap; at the slice where $y_{13} = 0$, therefore, and on the plane where $y_{12} = 0 = y_{13}$, the diagonal line $x_{12} = x_{13}$ maps configurations where all three agents have the same y coordinate, and agent 2 is on top of agent 3. These three graphs illustrate that pairwise obstacle functions (i.e., collision between 1 and 2, or collision between 1 and 3) define regions in the relative position space which are not isolated, and irrespectively of the agents' volume the origin of this space will always be a point common to all regions.

The distance-to-collision function is nonsmooth, not only at the origin, but anywhere the pair of closest neighbors changes. For these cases, we use Clarke's generalized (directional) derivative and gradient, which are defined as follows:

Definition 3 (Generalized derivative [18]): The generalized derivative of $f(x)$ in the direction v is

$$f^\circ(x; v) = \limsup_{\substack{y\to x\\ h\downarrow 0}} \frac{f(y + h\, v) - f(x)}{h} \ .$$

Definition 4 (Generalized gradient [18]): The generalized gradient of $f(x)$ at $x \in X$ is

$$\partial f(x) = \{\zeta \in X^* \mid f^\circ(x; v) \geq \langle \zeta, v\rangle\} \ ,$$

where X^* is the dual space of continuous linear functionals on X, and $\langle \zeta, v\rangle$ denotes the value of functional ζ at v.

In fact, when X is finite dimensional, and if $\Omega_f \subset X$ is the set of points where f is not differentiable, we can write

$$\partial f(x) = \mathrm{co}\left\{ \lim_{i\to\infty} \nabla f(x_i) : x_i \to x,\, x_i \notin \mathcal{M} \cup \Omega_f \right\}, \quad (10)$$

where co stands for the convex hull, $\mathcal{M}$ can be any set of measure zero, and x_i any sequence converging to x.

We define the finite sets

$$\Omega \triangleq \{p_i \in \mathbb{R}^n \mid 1 \leq i \leq N\}, \quad \Omega_i \triangleq \{p_j \in \Omega \mid j \neq i\} \ .$$

Set Ω includes all agent position vectors, and Ω_i just excludes the position of agent i. These two sets allow us to express the distance function of (9), that is, $d(q) \triangleq \min_{\substack{i,j\in\{1,\ldots,N\}\\ i\neq j}} \|q_{ij}\|$, in terms of the distance d_i,

$$d_i(p_i) \triangleq \min_{z\in\Omega_i} \|p_i - z\| = \min_{\substack{j=1,\ldots,N\\ i\neq j}} \|q_{ij}\| \ , \quad (11)$$

between $p_i \in \Omega$ and the rest of its groupmates as follows:

$$d(q) = \min_{1\leq i\leq N} d_i(p_i) = \min_{i=1,\ldots,N} \ \min_{\substack{j=1,\ldots,N\\ i\neq j}} \|q_{ij}\| \ . \quad (12)$$

With reference to the point-to-set distance d_i, we define

$$W(p_i) = \{z \in \Omega_i \mid \|p_i - z\| = d_i(p_i)\} \ ,$$

which identifies the nearest neighbor(s) of agent i. The indices of the nearest neighbors (the agents equidistant from i with the smallest distance from i) are contained in the sets

$$I_i = \big\{ j \in \{1, \ldots, N\} : p_j \in W(p_i) \big\} \ .$$

Existing literature [19, Theorem 1] allows us to state the following fact about the distance d_i, defined in (11) in terms of the Euclidean norm:

Corollary 1: For $p_i \neq p_j$, for all $j \in \{1, \ldots, N\}\backslash\{i\}$, (i.e., when $q_{ij} \neq 0$), $\partial d_i(p_i) = \mathrm{co}\{\partial \|q_{ij}\|, j \in I_i\}$.

Note that away from $q_{ij} = 0$, the generalized gradient of the norm of the relative position is the unit vector that points away from p_i and toward p_j, i.e., $\partial \|q_{ij}\| = \nabla \|q_{ij}\| = \frac{p_j - p_i}{\|q_{ij}\|}$.

The following statement practically follows from [18, Proposition 2.3.12] (the case of pointwise maxima)—but we were not able to locate it in literature in the general form stated below, so we provide a proof for it. Stronger versions of it exist for the case where the functions within the min are convex or regular [20].

Proposition 1 (Pointwise minima):

$$\partial \min_{i=1,\ldots,n} f_i(x) \subset \mathrm{co}\{\partial f_i(x) : i = 1,\ldots,n\} \ .$$

Proof:

$$\begin{aligned}\partial \min_{1\le i\le n} f_i(x) &= \partial\left(-\max_{1\le i\le n}\{-f_i(x)\}\right)\\ &= -\partial \max_{1\le i\le n}\{-f_i(x)\}\\ &\overset{\text{[18, Proposition 2.3.12]}}{\subset} -\mathrm{co}\left\{\partial\big(-f_i(x)\big) : 1\le i\le n\right\}\\ &= -\mathrm{co}\left\{-\partial f_i(x) : 1\le i\le n\right\}\\ &\overset{\text{[20, Proposition A.1]}}{=} \mathrm{co}\left\{\partial f_i(x) : 1\le i\le n\right\} \ .\end{aligned}$$

■

As a result, we can write

$$\partial d(q) \subset \mathrm{co}\{\partial\|q_{ij}\| : i \in \{1,\ldots,N\},\, j \in I_i\} \ ,$$

where the derivatives are taken with respect to q_{ij}. Away from points where there exist $i,j \in \{1,\ldots,N\}$ such that $q_{ij} = 0$, $\partial\|q_{ij}\|$ is a singleton (recall that gradients are taken with respect to q_{ij}, so in each case the derivative is that of the distance of a vector from the origin) and given (10),

$$\partial d(q) \subset \mathrm{co}\left\{\lim_{q'\to q} \nabla_q\|q_{ij}\|, i \in \{1,\ldots,N\},\, q_{ij} \neq 0,\, j \in I_i\right\}.$$

These convex hulls do not contain the zero vector away from $q_{ij} = 0$. This may become clearer with the following two-dimensional example.

Let $f(x,y) = \min\{|x|,|y|\}$, in which case we have $f_1(x,y) = |x|$, and $f_2(x,y) = |y|$. The graph of $f(x,y)$ is shown in Fig. 2(a).

The gradients of f_1 and f_2 away from $x = 0$ and $y = 0$ respectively, are

$$\nabla|x| \overset{x\neq 0}{=} \begin{cases}(1,0), & x>0\\ (-1,0), & x<0\end{cases}, \quad \nabla|y| \overset{y\neq 0}{=} \begin{cases}(1,0), & y>0\\ (-1,0), & y<0\end{cases} .$$

Therefore, away from the axes and the lines $x = \pm y$, the generalized gradient of f will be a singleton:

$$\nabla f(x,y) = \begin{cases}(0,1), & 0<x<y\\ (1,0), & 0<y<x\\ (-1,0), & -y<x<0\\ (0,1), & x<-y<0\\ (0,-1), & x<y<0\\ (-1,0), & y<x<0\\ (1,0), & 0<x<-y\\ (0,-1), & 0<-y<x.\end{cases}, \quad x\neq \pm y,\, x,y\neq 0 \ .$$

It can be seen (see Fig. 2(b)) that along the lines $x = \pm y$ for $x, y \neq 0$, ∂f is a convex set that does not contain the origin.

The distance function $d(q)$ is essentially a generalization of $f(x,y)$ in multiple dimensions and behaves very similarly in terms of its (generalized) derivatives. A notable property of ∂d is the following:

Lemma 1: Let $d(q)$ be defined by (12). Then for $q \neq 0$, and for any z, w such that $z \in \partial d(q)$ and $w \in -\partial d(q)$, we have $\langle z, w\rangle \le 0$. In addition, $\langle z, w\rangle = 0$ only when $z \in \delta\partial d$ and $w \in \delta\partial d$.

Proof: The generalized gradient of $d(q)$ is the convex hull of unit vectors e_i along different coordinate directions q_i. Any $z \in \partial d(q)$ is thus written as $z = \sum a_i e_i$, with $\sum a_i = 1$ and $a_i > 0$, while $w = \sum b_i(-e_i)$, with $\sum b_i = 1$ and $b_i > 0$, where i takes values in a finite set $\{1,\ldots,k\}$ for some k. Therefore, $\langle z, w\rangle = \sum_{i,j\in\{1,\ldots,k\}}(-a_i b_j)\langle e_i, e_j\rangle$, where we note that $\langle e_i, e_j\rangle = 1$ if $i = j$ and zero otherwise. Since $a_i, b_j \ge 0$, the sum is negative semidefinite. For $\langle z, w\rangle = 0$ we need to have $(-a_i b_i)\langle e_i, e_i\rangle = 0 \Rightarrow a_i b_i = 0$. For all i therefore, either a_i or b_i are zero, implying that in this case both z and w are on the boundary of ∂d and $-\partial d$, respectively.

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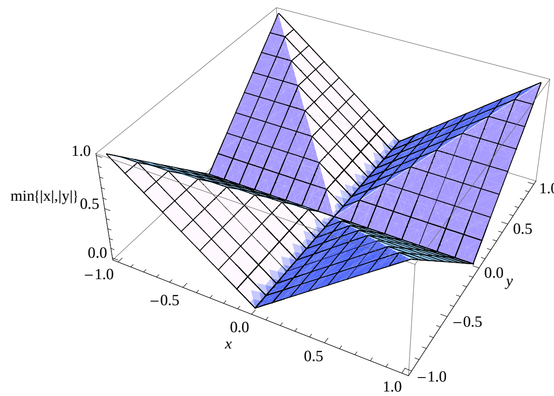
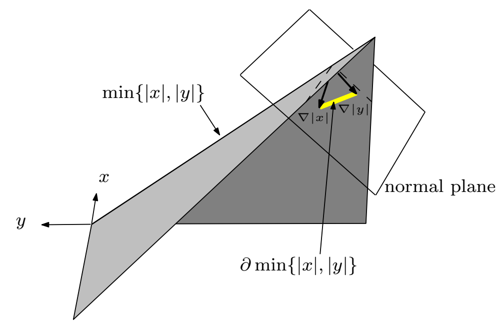


(a) Graph of $\min\{|x|,|y|\}$



(b) Generalized gradient along the $x = y$ line

Fig. 2. The $\min\{|x|,|y|\}$ function and its generalized derivative. Fig. 2(a) shows that the function has minima along the x and y axes. Along the lines $x = y$ and $x = -y$ the function is not differentiable. Fig. 2(b) illustrates why the generalized gradient along the $x = \pm y$ lines does not contain zero: $(0,0) \notin \mathrm{co}\{(0,1),(1,0)\}$! There is no positive λ for which $(0,0) = \lambda(0,1) + (1-\lambda)(1,0)$.

For functions expressed as pointwise minima (the case of maxima can be treated similarly) where Proposition 1 applies, we know the following:

Lemma 2 ([20]): The origin is contained in the interior of the convex hull of a set of n arbitrary vectors $\{v_i \in \mathbb{R}^m,\, i = 1,\ldots,n\}$ iff there exists a v_i such that for all $w \in \mathbb{R}^m$, $\langle w, v_i\rangle > 0$.

IV. Properties of the Proposed Construction

The analysis of this section focuses on the properties of the gradient of φ. Specifically, it investigates the nature of its critical points as a function of parameter κ which is shown to control the size of $\mathcal{Q}_0$.

The discussion is organized in two parts. Section IV-A introduces the mathematical preliminaries required to analyze the behavior of the nonsmooth critical points of φ. These points appear at locations where the distance function d is not differentiable, and some of the properties of the generalized gradient of the latter are revealed in Section III-C. Section IV-B makes the formal statements that establish the convergence properties of the gradient field of the navigation function. The proofs for these statements analyze two alternative cases: the distance function being differentiable at the critical point, and the critical point being a nonsmooth one.

A. Nonsmooth analysis preliminaries

When the right-hand side of the differential equation (3) is only piecewise continuous, the solution $p(t)$ (and therefore $q(t)$) for $t \geq 0$, is understood in terms of a Filippov solution to the differential inclusion

$$\dot{q} \in F\big(q(t)\big), \quad q(t) \in \mathcal{Q} \ , \tag{13}$$

where $F(\cdot)$ is the Filippov set-valued map. Since in this case $\mathcal{Q}$ is finite dimensional, $F(\cdot)$ is given by

$$F(q) = \overline{\text{co}} \left\{ \lim_{k\to\infty} Bu\big(q[k]\big) : q[k] \to q, q[k] \notin \mathcal{M} \right\} \ ,$$

where $\overline{\text{co}}$ denotes the closure of the convex hull, $q[k]$ is any sequence of points at which φ is differentiable, B is the incidence matrix of the formation graph, and $\mathcal{M}$ can be any set of measure zero. Such Filippov solutions are absolutely continuous curves $q : [0, \infty) \to \mathcal{F}$, which satisfy (13) for almost all $t \in [0, \infty)$. For the existence properties of Filippov solutions, we refer to [21], [22].

With φ being continuous, positive definite and radially unbounded, its level sets will be compact and invariant with respect to relative position trajectories.

Consider a function $f : \mathcal{Q} \to \mathbb{R}$, and let $f_1, \ldots, f_m : \mathcal{Q} \to \mathbb{R}$ be a set of continuous functions. If it happens that $I(q) \triangleq \{i \in \{1, \ldots, m\} \mid f(q) = f_i(q)\} \neq \varnothing$, $\forall q \in \mathcal{Q}$, then f is called a *continuous selection* of functions $f_1, \ldots, f_m$, and $I(q)$ is the *index set* of f at q. The set of all continuous selection functions of $f_1, \ldots, f_m$ is denoted $\mathsf{CS}(f_1, \ldots, f_m)$ [23].

Definition 5 (Nonsmooth critical point [24]): If $\bar{q}$ is a critical point for $f \in \mathsf{CS}(f_i, i \in I)$, then there exist real numbers λ_i, for $i \in I(\bar{q})$, with $\Sigma_{i\in I(\bar{q})}\lambda_i \mathrm{d}f_i(\bar{q}) = 0$, $\Sigma_{i\in I(\bar{q})}\lambda_i = 1$, and $\lambda_i \geq 0$.

Here, the symbol d denotes the differential of a function, which—although not different for practical purposes from the gradient—it is reserved here for the discussion of cases that involve (nonsmooth) continuous selection functions.

It is known [23] that if f is a continuous selection of differentiable functions then it is locally Lipschitz and its generalized gradient is given by $\partial f(q) = \text{co}\{\nabla f_i(q) \mid i \in \hat{I}(q)\}$, where $\hat{I}(q) = \{i \mid q \in \overline{\text{int}}\{q \mid f(q) = f_i(q)\}\}$ with $\overline{\text{int}}$ denoting the closure of the interior of a set.[2]

[2]That latter operation practically regularizes the domain of f, discarding subsets of measure zero and thus giving it dimensional homogeneity. The resulting index set is represented by $\hat{I}$.

Definition 6 (Nonsmooth non-degenerate critical point [23]): Let $f \in \mathsf{CS}(f_1, \ldots, f_m)$ with $f_1, \ldots, f_m : \mathcal{Q} \to \mathbb{R}$ be twice differentiable functions. A critical point $q_0 \in \mathcal{Q}$ of f is called non-degenerate if the following two conditions hold:

(ND1) For each $i \in \hat{I}(q_0)$, the set of differentials $\left\{\mathrm{d}f_j(q_0) | j \in \hat{I}(q_0) \setminus \{i\}\right\}$ is linearly independent, and

(ND2) The second differential $\mathrm{d}^2 L(\cdot, \hat{\lambda})(q_0)$ of the Lagrangian $L(q, \lambda) \triangleq \sum_{i\in \hat{I}(q_0)} \lambda_i f_i(q)$ is regular on $\hat{T}(q_0) = \cap_{i\in \hat{I}(q_0)} \text{kern } \mathrm{d}f_i(q_0)$, for $\hat{\lambda}_i \in \mathbb{R}$ such that $\mathrm{d}L(\cdot, \hat{\lambda})(q_0) = 0$, $\sum_{i\in\hat{I}(q_0)} \hat{\lambda}_i = 1$, and $\hat{\lambda}_i \geq 0$ $\forall i \in \hat{I}(q_0)$.

For our case, the linear subspace $\hat{T}(q_0)$ at q_0 can take the form $\hat{T}(q_0) = \left\{\zeta \in \mathbb{R}^n : \zeta^T \mathrm{d}f_i(q_0) = 0, \ \forall i \in \hat{I}(q_0)\right\}$. Furthermore, for the second condition we need every $\nu \in \hat{T}(q_0)$ to satisfy $\nu^T \, \mathrm{d}^2 L(\cdot, \hat{\lambda})(q_0) \, \nu \neq 0$.

B. Properties of critical points

The proofs for all statements are found in the Appendix. The following proposition states that the destination configuration q_d is a non-degenerate minimum of φ:

Proposition 2: The destination point, q_d, is a non-degenerate minimum of $\bar{\varphi}$.

We can also show that there are no critical points on the workspace boundary $\delta\mathcal{Q}$:

Proposition 3: All critical points of $\bar{\varphi}$ are in $\text{int}\mathcal{Q}_2$.

From this point on, the critical point properties are proved for φ, for reasons of analytical simplicity only, since the critical points of $\bar{\varphi}$ and φ coincide are are of the same nature [7, Proposition 2.7]. We next show that critical points can be pushed away from $\mathcal{Q}_2$, and arbitrarily close to the collision configurations in $\mathcal{Q}_0$, by varying the tuning parameter κ:

Proposition 4: For every $\epsilon > 0$ for which $q_d \notin \text{cl}\mathcal{B}_q(\epsilon)$, there exists a lower bound for κ above which there are no critical points of φ in $\mathcal{Q}_2(\epsilon)$.

The next result ensures that with an appropriate choice of parameter, the critical points that have been pushed toward the obstacles are not local minima:

Proposition 5: Suppose the formation specification is valid. Then there exists an $\epsilon_0 > 0$ such that φ has no local minimum in $\mathcal{Q}_0$ as long as $\epsilon < \epsilon_0$.

To show that the *smooth* critical points of φ are non degenerate, we first use a lemma from [7], which asserts that the non-singularity of a linear operator follows from the sign definiteness on orthogonal subspaces of its quadratic form:

Lemma 3: ([7, Lemma 3.1]) Let $\mathbb{R}^n = \mathcal{P} \oplus \mathcal{N}$ and let the symmetric matrix $A \in \mathbb{R}^{n\times n}$ define a quadratic form on $\mathbb{R}^n$, $\xi(v) \triangleq v^T A v$. If $\xi|_{\mathcal{P}}$ (the restriction of ξ in $\mathcal{P}$) is positive definite and $\xi|_{\mathcal{N}}$ is negative definite, then A is non-singular and $\text{index}(A) = \dim(\mathcal{P})$.

Now assume that φ is (twice) differentiable at q. Let $\xi_q(v)$ denote $v^T \nabla^2 \varphi(q) v$, where $q \in \mathcal{Q}_0(\epsilon)$, and v is a vector in the tangent space T_q of $\mathcal{Q}_0(\epsilon)$ at q. Then the remaining nonnegative eigenvalues of the Hessian of φ are all positive.

Proposition 6: (cf. [7, Proposition 3.6]) Assume that φ is twice differentiable at q, where $q \in \mathcal{Q}_0(\epsilon)$ is a critical point of φ. Then there exists an $\epsilon_2 > 0$ such that for every $\epsilon <$

ϵ_2, there is a direct sum decomposition $T_q = \mathcal{P}_q \oplus \mathcal{N}_q$, for which $\xi_q|_{\mathcal{P}_q}$ is positive definite, $\xi_q|_{\mathcal{N}_q}$ is negative definite, and $\dim(\mathcal{P}_q) = 1$.

Thus, for an appropriately small ϵ (read: sufficiently large κ), a smooth critical point of φ is non-degenerate, since the Hessian at that point has a single positive eigenvalue while all other are negative. Now consider the case where φ is nonsmooth at q:

Proposition 7: Let φ be nondifferentiable at $q \in \mathcal{Q}_0(\epsilon)$, and q be a nonsmooth critical point. Then there exists an ϵ_3 such that for all $\epsilon < \epsilon_3$, $\mathrm{d}^2 L(\cdot, \hat{\lambda})(q)$ is regular on $\hat{T}(q)$.

Together, Propositions 2 through 7 establish that when appropriately tuned, and irrespectively of whether or not it is differentiable at a critical point, φ defined based on (5)-(6)-(7) all of its critical points are non-degenerate and the function has a single minimum at q_d—all other critical points are saddles. As it turns out, what affects the nature of the critical points of φ is the selection of parameters κ and ζ, as well as the formation specifications. Ideally, one would like to have a) a desired formation configuration away from collision configurations, b) a relatively high value for ζ i.e., the derivative of the obstacle function at the boundary of the free space, and c) a sufficiently large exponent κ. Once φ is tuned, its negated gradient field as input can yield almost global asymptotic convergence of (13) to q_d. To see that, tet us first design the right-hand side of (13) by setting

$$(B^{\mathsf{T}} \otimes I)\, u = -\mathrm{Ln}(\partial\varphi)(q) \ , \tag{14}$$

where $\mathrm{Ln}(S)(q)$ is a set valued map that assigns to each subset of S the set of least-norm elements in the closure of S [22]; in cases where S is convex and closed (as for $S = \partial\varphi$), $\mathrm{Ln}(S)(q)$ maps to a singleton which is the orthogonal projection of the zero vector on S. The right hand side of (13) takes the form of (14) if the agent inputs are chosen as

$$u = -(B^{\mathsf{T}} \otimes I)^{\dagger}\, \mathrm{Ln}(\partial\varphi)(q) \ , \tag{15}$$

where $(\cdot)^{\dagger}$ denotes the Moore-Penrose generalized inverse, and B is the incidence matrix of the formation graph $\mathcal{G}$. With this choice of control inputs

$$F(q) = -\partial\varphi(q) \ ,$$

and (13) takes the form of a nonsmooth generalized gradient flow [22].

In our case, φ is locally Lipschitz and regular because it is a pointwise maximum function [18, Proposition 2.3.12]. We can therefore bring to bear a strong result that directly applies to nonsmooth generalized gradient flows [22, Proposition 11]:

Proposition 8 ([22]): Let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and regular. Then the strict minimizers of V are strongly stable equilibria of the nonsmooth gradient flow of V. Furthermore, if the level sets of V are bounded, then the solutions of the nonsmooth gradient flow asymptotically converge to the set of critical points of V.

In our case, taking $\varphi(q)$ as V, we know that the level sets of φ are bounded in $\mathcal{Q}$. The convergence of the closed loop system (13)–(15) therefore follows from a direct application of Proposition 8.

V. Simulations and Experiments

A. Numerical results

We assess the behavior of the closed loop system in the neighborhood of a suspected saddle through numerical simulations. In this scenario we set $c_{12} = (0.1, 0.1)$ m and $c_{23} = (0.2, 0.2)$ m, so that the agents are expected to align along a 45 degree line, with agent 2 in the middle, agent 1 at the top right, and agent 3 at the bottom left. We choose initial positions for the agents so that they are all along the 45 degree line, but at the “wrong” place: agent 1 is at the bottom left, agent 2 is at the top right, and agent 3 is in the middle. The agents have to swap relative positions to reach the desired configuration. Intuition suggests that the symmetry of the initial configuration would force the agents to a deadlock; indeed our simulations show (these results are not included due to space limitations) that many initial configurations with the agents arranged along the same 45° degree line in that order, converge to the same attractor on the line, $x_{12} = y_{12} = -0.4$ and $x_{23} = y_{23} = 0.2$ suggesting that this line is in the attraction region of the critical point.

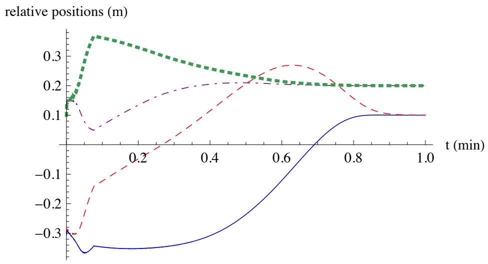
Figure 3 shows what happens when the formation is initialized slightly off the region of attraction of the saddle. Agents are able to swap positions, minimize their goal function asymptotically, and achieve their desired configuration. What this test demonstrates is that the attractor at $x_{12} = y_{12} = -0.4$, $x_{23} = y_{23} = 0.2$, is in fact unstable, i.e., a saddle configuration. These numerical simulations were performed with MATLAB™, on a 32 bit Intel Core 2 Duo processor (each core running at 2GHz), with 3GB of RAM. In this computing environment, running time varies with the number of agents as shown in Table I. Simulation results for the paths of six agents initially arranged on a line and having to switch positions, are shown in Fig. 7(a). The original and desired configurations for these agents are shown in Fig. 6, whereas the time evolution of this formation's goal function is shown in Fig. 7(b).

TABLE I
Computation Time

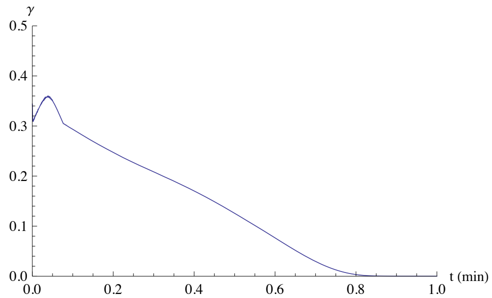
	No of agents			
	3	4	5	6
Time (sec)	28.74	55.45	93.62	320.24

B. Experimental results

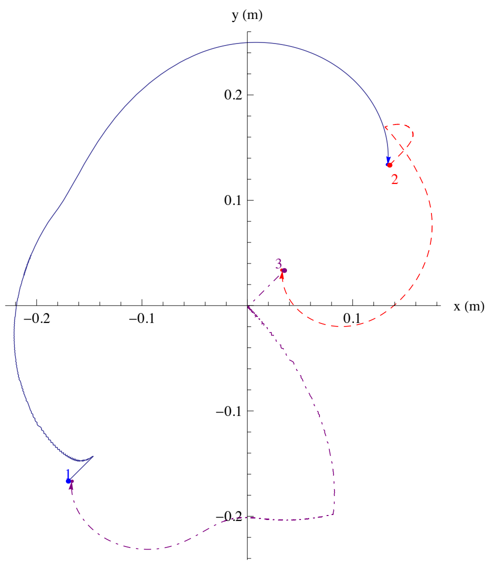
A collection of mobile robotic platforms shown in Fig. 4 is used to validate the simulation results through experiments. The scenario is similar, with the objective being once again to reposition the robots in the same order along a 45° line, but now the desired relative positions are different: $c_{12} = (0.75, 0.75)$ m, $c_{23} = (1.5, 1.5)$ m. The experimental results are shown in Fig. 5. Besides the obvious difference in the size of the position errors at the end of the time interval of the test, another notable difference compared to the simulation results is the jittering paths of the robots. Both of these observations are attributed to three main reasons: a) the implementation of the (holonomic) control law (15) on a nonholonomic



(a) Evolution of relative positions (q_{ij}).



(b) Evolution of the goal function γ.



(c) Agent paths.

Fig. 3. Simulation of a formation control maneuver where all agents have to switch positions, starting *close* but not in the attraction region of a saddle. The formation specifications are $c_{12} = (0.1, 0.1)$ m and $c_{23} = (0.2, 0.2)$ m. In the initial configuration, $q_{12}(0) = (-0.305, -0.3)$ m and $q_{23}(0) = (0.1, 0.1)$ m. In Fig. 3(a) the continuous curve shows the evolution of $x_1 - x_2$, the dashed curve shows that of $y_1 - y_2$, the dot-dashed curve depicts $x_2 - x_3$, and the thick dotted one gives $y_2 - y_3$. Figure 3(b) indicates that the desired formation is reached since the goal function γ converges to zero. Figure 3(c) shows the paths of the three agents as they switch positions to reconfigure themselves into the desired formation.

system by means controlling each agent's orientation so that it aligns itself with the negated gradient direction resulting from (15), b) the quantization errors induced by the finite input velocities available for these platforms, and c) the existence of a "deadzone" region of input values around zero, which prevented the platforms from moving with arbitrarily small linear and rotational speeds.

Fig. 4. A three-robot group used in the formation control experiments. The robots localize themselves using a motion capture system. Since their motion is subject to nonholonomic constraints, the control law is implemented by projecting the negated gradient direction on their feasible direction of motion.

VI. Conclusion

Existing generalizations of the navigation function approach [6] to a multi-agent setting, typically employ the product of positive semidefinite functions as a metric of the distance of the system from collision configurations. We indicate that following the same proof techniques that have appeared in literature, may fail to establish the non-degeneracy of the function's critical points. The paper suggests an alternative construction for the potential function for which the non-degeneracy of critical points, and the disappearance of local minima with appropriate tuning, can be analytically shown. The new construction is a nonsmooth positive definite function, and the proof techniques are based on nonsmooth analysis, and control theory for dynamical systems expressed in the form of (finite-dimensional) differential inclusions.

Appendix

A. *Proof of Proposition 2: destination is non-degenerate*

At configurations where d is differentiable, we can write

$$\nabla\bar{\varphi} = \frac{1}{\beta^{2/\kappa}}\left(\beta^{1/\kappa}\nabla\gamma_d - \gamma_d\frac{1}{\kappa}\beta^{1/\kappa-1}\nabla\beta\right) \ . \qquad (16)$$

The Hessian of $\bar{\varphi}$, on the other hand, is

$$\nabla^2\bar{\varphi} = \frac{1}{\beta^{\frac{4}{\kappa}}}\Big[\beta^{\frac{2}{\kappa}}\Big(\beta^{\frac{1}{\kappa}}\nabla^2\gamma_d + \nabla\beta^{\frac{1}{\kappa}}\nabla\gamma_d{}^T - \nabla\gamma_d(\nabla\beta^{\frac{1}{\kappa}})^T - \gamma_d\nabla^2\beta^{\frac{1}{\kappa}}\Big) - 2\beta^{\frac{1}{\kappa}}(\beta^{\frac{1}{\kappa}}\nabla\gamma_d - \gamma_d\nabla\beta^{\frac{1}{\kappa}})(\nabla\beta^{\frac{1}{\kappa}})^T\Big] \ .$$

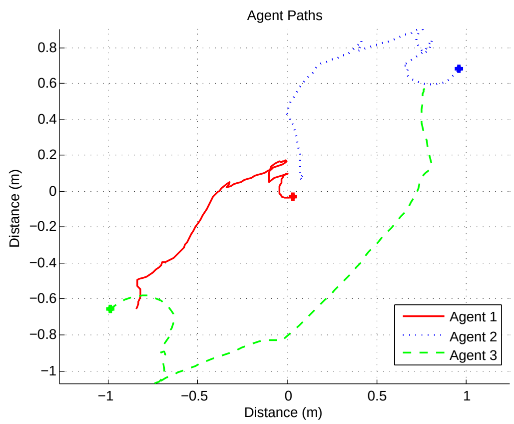
At q_d, we have $\nabla\gamma_d(q_d) = 0$ and $\gamma_d(q_d) = 0$. Thus with $\nabla\bar{\varphi}(q_d) = 0$, the Hessian reduces to $\nabla^2\bar{\varphi} = \frac{\nabla^2\gamma_d}{\beta^{\frac{3}{\kappa}-2}}$, and since $\nabla^2\gamma_d = 2I$ we have $\nabla^2\bar{\varphi} = \frac{2}{\beta^{\frac{3}{\kappa}-2}}I$, where I here denotes the $\frac{N(N-1)}{2}$-dimensional identity matrix. With $\beta(q_d) > 0$, it follows that q_d is a (smooth) non-degenerate critical point.

When the distance function is not differentiable at q_d, then according to Definition 5, since q_d is a nonsmooth critical point, we have

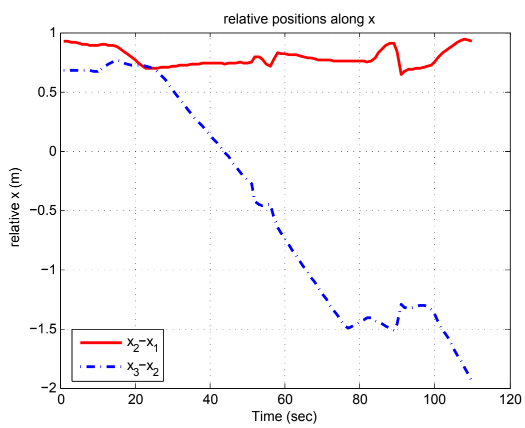
$$\sum_i \lambda_i \mathrm{d}\bar{\varphi}_i = \sum_i \lambda_i \frac{\nabla\gamma_d - \frac{\gamma_d}{\kappa\beta}\mathrm{d}\beta_i}{\beta^{\frac{1}{\kappa}}} = 0 \ , \qquad (17)$$

where the subscript i has been dropped from β_i since all β_i involved in the sum are equal at the critical point. Because of (17), the second differential of the Lagrangian $L(q, \lambda)$ when evaluated at a critical point simplifies to

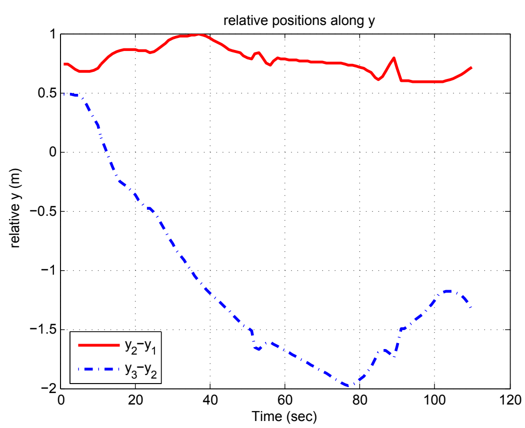
$$\sum_i \lambda_i \mathrm{d}^2\bar{\varphi} = \sum_i \lambda_i \frac{\beta^{\frac{1}{\kappa}}\nabla^2\gamma_d - \gamma_d \mathrm{d}^2\beta_i{}^{\frac{1}{\kappa}}}{\beta^{\frac{2}{\kappa}}} \ .$$



(a) Agent Paths



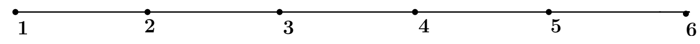
(b) relative positions along x



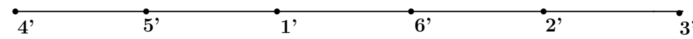
(c) relative positions along y

Fig. 5. Experimental results showing the switching of relative positions of three agents in a straight line. The dots mark the final position of each agent.

Since $\nabla^2\gamma_d = 2I$, $\gamma_d(q_d) = 0$, and $\sum_i \lambda_i = 1$, the second differential of the Lagrangian reduces to $2\beta^{\frac{-1}{\kappa}} I \neq 0$ and is therefore regular at q_d.

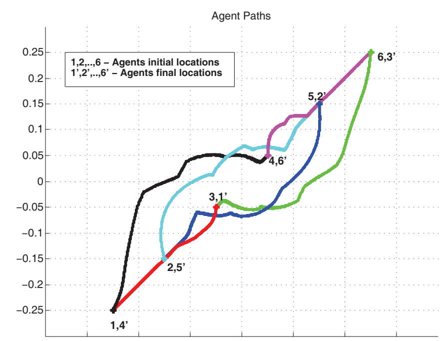


(a) Initial configuration of the agents.

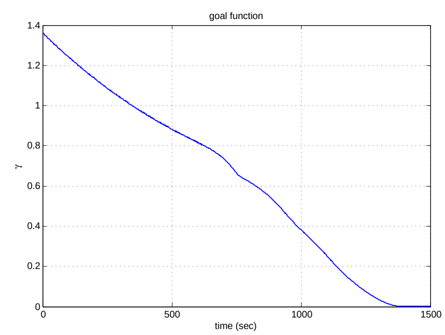


(b) Desired configuration of the agents.

Fig. 6. Initial and final configurations for the 6-agent simulation test



(a) Agent paths



(b) Goal function

Fig. 7. Simulation of a 6-agent formation control maneuver where the agents switch positions in accordance with Fig. 6. The agents begin on a 45^o straight line in the neighborhood of a saddle configuration. The relative distances between successive agents is $(0.1, 0.1)$. Figure 7(a) shows the agent paths, with unprimed numbers representing the initial configuration and primed numbers the final configuration, while Fig. 7(b) shows the evolution over time of the goal function.

B. Proof of Proposition 3: no critical points on workspace boundary

Assume first that $d(q)$ is differentiable at a critical point q. Recall (16) and note that note that on $\partial\mathcal{F}$ we have $\beta = 0$, while $\gamma_d \neq 0$, $\nabla\beta \neq 0$, the latter due to a choice of $\zeta > 0$. Therefore, as $q \to \partial\mathcal{F}$, $\|\nabla\bar{\varphi}\| \to \infty$ while $\angle\nabla\bar{\varphi} \to \angle\nabla\beta$, establishing the transversality of $\nabla\bar{\varphi}$ on the boundary.

In the case where $d(q)$ is not differentiable at critical point q, and in view of (17), we can see that using the chain rule [18, Theorem 2.3.9(ii)] on $\partial\beta$, we can write the generalized gradient of $\bar{\varphi}$ as

$$\partial\bar{\varphi} = \frac{1}{\kappa\beta^{\frac{1}{\kappa}+1}}\left(\kappa\beta\nabla\gamma_d - \gamma_d\,\frac{\mathrm{d}\beta}{\mathrm{d}d}\,\partial d\right) \quad . \tag{18}$$

As one approaches collision configurations, $\beta \to 0$, (18) suggests that $\partial\bar{\varphi} \to -\gamma_d(\kappa\beta^{\frac{1}{\kappa}+1})^{-1}\,\frac{\mathrm{d}\beta}{\mathrm{d}d}\,\partial d$, which does not contain the zero vector. Therefore, q cannot be a nonsmooth critical point.

C. Proof of Proposition 4: critical points can be pushed toward collision configurations

Let q be a critical point of φ, and consider first the case where $d(q)$ is differentiable at q. Since q is a critical point $\nabla\varphi = 0$, and from (16) it follows that at this configuration $\beta\nabla\gamma = \gamma\nabla\beta$. Substituting for γ from (6) we get

$$\beta\kappa{\gamma_d}^{\kappa-1}\,\nabla\gamma_d = {\gamma_d}^{\kappa}\,\nabla\beta \Rightarrow \beta\kappa\,\nabla\gamma_d = \gamma_d\,\nabla\beta \ .$$

Taking norms on both sides yields $\kappa\beta\,\|\nabla\gamma_d\| = \gamma_d\|\nabla\beta\|$, from which we derive a sufficient condition for q *not* to be a critical point:

$$\kappa > \frac{\gamma_d\|\nabla\beta\|}{\beta\,\|\nabla\gamma_d\|} \ . \tag{19}$$

The gradient of the obstacle function (7) is

$$\nabla\beta = \frac{2a(2d+1)(d+d^2-r)}{\mu\, e^{(d+d^2-r)^2}-a}\nabla d \ ,$$

where ∇d is the gradient of the distance function (9). (Recall that we have assumed $d(q)$ to be differentiable at q.) We have

$$\|\nabla\beta\| = \left|\frac{2a(2d+1)(d+d^2-r)}{\mu\, e^{(d+d^2-r)^2}-a}\right|\,\|\nabla d\| \ .$$

It can be verified that the right hand side of (19) (as long as the formation specification is valid) is bounded in $\mathcal{Q}_2$. In fact, since $\nabla\beta$ is upper bounded everywhere in $\mathcal{Q}_2$ and β attains a minimum of ϵ at the boundary of $\mathcal{Q}_2$, it follows that the ratio $\frac{\|\nabla\beta\|}{\beta}$ is upper bounded in $\mathcal{Q}_2$. In addition, since γ_d and $\|\nabla\gamma_d\|$ are continuous functions that must attain their extremum points in the closure of $\mathcal{Q}_2$, and given that as $q\to q_d$, $\frac{\gamma_d}{\|\nabla\gamma_d\|}\to 0$, the bound for κ on the right hand side of (19) is finite anywhere in the closure of $\mathcal{Q}_2$.

Now consider the case where $d(q)$ be nondifferentiable at q. Then a necessary condition for q to be a critical point is $0\in\partial\varphi \overset{(18)}{\Rightarrow} 2\kappa\beta\left(\gamma_d\frac{\partial\beta}{\partial d}\right)^{-1}(q-q_d)\in\partial d$. The left hand side of this inclusion can be turned into an arbitrarily large vector by an appropriate assignment to κ: (i) away from destination and collision configurations β and $q-q_d$ are lower bounded by ϵ and $\sqrt{\epsilon}$, respectively, and (ii) in any domain containing q_d, the terms in the denominator, γ_d and $\frac{\partial\beta}{\partial d}$ are upper bounded. On the other hand, ∂d is always lower and upper bounded. Thus, for a sufficiently large κ the critical points of φ can not lie inside Q_2.

D. Proof of Proposition 5: Near-collision critical points are not minima

Assume first that q is a smooth critical point of φ. The Hessian of φ at q is generally written as

$$\nabla^2\varphi = \frac{1}{\beta^3}\left[\beta\left(\beta\nabla^2\gamma - \gamma\nabla^2\beta + \nabla\beta\,\nabla\gamma^T - \nabla\gamma\,\nabla\beta^T\right)\right.$$
$$\left. -2(\beta\nabla\gamma-\gamma\nabla\beta)\nabla\beta^T\right] \ . \tag{20}$$

Since q is a critical point, we have $\beta\nabla\gamma-\gamma\nabla\beta=0$, so with (6), (20) reduces to

$$\nabla^2\varphi = \frac{1}{\beta^2}\left[\beta\nabla^2{\gamma_d}^{\kappa} - {\gamma_d}^{\kappa}\nabla^2\beta\right] = \frac{{\gamma_d}^{\kappa-2}}{\beta^2}\times$$
$$\left[\kappa\beta\left(\gamma_d\nabla^2\gamma_d + (\kappa-1)\nabla\gamma_d\nabla{\gamma_d}^T\right) - {\gamma_d}^2\nabla^2\beta\right] \ . \tag{21}$$

From $\beta\nabla\gamma-\gamma\nabla\beta=0$, after plugging (6) we get

$$\kappa\beta\nabla\gamma_d = \gamma_d\nabla\beta \ , \tag{22}$$

and taking the outer product on both sides of (22),

$$(\kappa\beta)^2\nabla\gamma_d\nabla{\gamma_d}^T = {\gamma_d}^2\nabla\beta\nabla\beta^T \ . \tag{23}$$

Equation (23) can be solved for $\nabla\gamma_d\nabla{\gamma_d}^T$, and after substitution in (21) we get

$$\nabla^2\varphi = \frac{\gamma_d^{\kappa-1}\left[\kappa\beta\nabla^2\gamma_d + \frac{(\kappa-1)\gamma_d}{\kappa\beta}\nabla\beta\nabla\beta^T - \gamma_d\nabla^2\beta\right]}{\beta^2}. \tag{24}$$

From (22) again, by taking norms on both sides, we have $\kappa\beta = \frac{\gamma_d\|\nabla\beta\|}{\|\nabla\gamma_d\|}$, which we can substitute in (24) to get

$$\nabla^2\varphi = \frac{{\gamma_d}^{\kappa}}{\beta^2}\left[\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|}\nabla^2\gamma_d + \left(1-\frac{1}{\kappa}\right)\frac{1}{\beta}\nabla\beta\nabla\beta^T - \nabla^2\beta\right].$$

Let $\tilde{v}$ be any unit vector orthogonal to $\nabla\beta$. Then the quadratic form $\tilde{v}^T\left(\nabla^2\varphi\right)\tilde{v}$ evaluates

$$\tilde{v}^T\left(\nabla^2\varphi\right)\tilde{v} = \frac{{\gamma_d}^{\kappa}}{\beta^2}\tilde{v}^T\left(2I\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|} - \nabla^2\beta\right)\tilde{v} \ , \tag{25}$$

where I is the $\frac{N(N-1)}{2}$-dimensional identity matrix. For the right hand of (25) to be negative, it suffices to have

$$\max\left\{2\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|}\right\} - \min\left\{\lambda\left(\nabla^2\beta\right)\right\} < 0 \ , \tag{26}$$

where $\lambda(\cdot)$ denotes a matrix eigenvalue. Recall (7) and treat β as a function of two (dependent) variables $x_1 \triangleq d$ and $x_2 \triangleq d^2$: $\beta = \beta(x_1,x_2)$. The first partial derivative of β with respect to q can be written

$$\nabla\beta = \frac{\partial\beta}{\partial x_1}\frac{\partial x_1}{\partial q} + \frac{\partial\beta}{\partial x_2}\frac{\partial x_2}{\partial q} \ ,$$

while the second partial derivative is

$$\nabla^2\beta = \frac{\partial^2\beta}{\partial {x_1}^2}\left[\frac{\partial x_1}{\partial q}\left(\frac{\partial x_1}{\partial q}\right)^T\right] + \frac{\partial\beta}{\partial x_1}\left[\frac{\partial^2 x_1}{\partial q^2}\right]$$
$$+ \frac{\partial^2\beta}{\partial {x_2}^2}\left[\frac{\partial x_2}{\partial q}\left(\frac{\partial x_2}{\partial q}\right)^T\right] + \frac{\partial\beta}{\partial x_2}\left[\frac{\partial^2 x_2}{\partial q^2}\right] \ .$$

With reference to (7) we have,

$$\frac{\partial\beta}{\partial x_1} = \frac{\partial\beta}{\partial x_2} = \frac{2a(x_1+x_2-r)}{\mu\, e^{(x_1+x_2-r)^2}-a} > 0 \ ,$$

for $x_1+x_2-r>0$ and

$$\frac{\partial^2\beta}{\partial {x_1}^2} = \frac{\partial^2\beta}{\partial {x_2}^2} =$$
$$\frac{2a[\mu\left(4r(x_1+x_2) - 2(x_1+x_2)^2 - 2r^2 + 1\right)e^{(x_1+x_2-r)^2} - a]}{\left(a-\mu e^{(x_1+x_2-r)^2}\right)^2},$$

which for $(x_1 + x_2) \to r^+$, in the region where the critical point is expected, converges to $\frac{2a}{\mu - a} > 0$, for $a < \mu$.

To determine $\min\left\{\lambda\left(\nabla^2\beta\right)\right\}$, we first note that since the partial derivatives of β have shown to be positive, we can write

$$\begin{aligned}\min\left\{\tilde{v}^T\nabla^2\beta\tilde{v}\right\} = \frac{\partial^2\beta}{\partial {x_1}^2}\min\left\{\tilde{v}^T\left[\frac{\partial x_1}{\partial q}\left(\frac{\partial x_1}{\partial q}\right)^T\right]\tilde{v}\right\} \\ + \frac{\partial\beta}{\partial x_1}\min\left\{\tilde{v}^T\left[\frac{\partial^2 x_1}{\partial q^2}\right]\tilde{v}\right\} \\ + \frac{\partial^2\beta}{\partial {x_2}^2}\min\left\{\tilde{v}^T\left[\frac{\partial x_2}{\partial q}\left(\frac{\partial x_2}{\partial q}\right)^T\right]\tilde{v}\right\} \\ + \frac{\partial\beta}{\partial x_2}\min\left\{\tilde{v}^T\left[\frac{\partial^2 x_2}{\partial q^2}\right]\tilde{v}\right\} \quad . \quad (27)\end{aligned}$$

The first and third term in (27) involve rank-one matrices made of the same vector, and thus their minimum eigenvalue is zero. With this observation, (27) implies

$$\begin{aligned}\min_{\mathcal{Q}_0}\left\{\tilde{v}^T\nabla^2\beta\tilde{v}\right\} \geq \frac{\partial\beta}{\partial x_1}\min_{\mathcal{Q}_0}\left\{\tilde{v}^T\frac{\partial^2 x_1}{\partial q^2}\tilde{v}\right\} \\ + \frac{\partial\beta}{\partial x_2}\min\left\{\tilde{v}^T\frac{\partial^2 x_2}{\partial q^2}\tilde{v}\right\} \quad ,\end{aligned}$$

and given that $\frac{\partial\beta}{\partial x_1} = \frac{\partial\beta}{\partial x_2}$,

$$\min_{\mathcal{Q}_0}\left\{\tilde{v}^T\nabla^2\beta\tilde{v}\right\} \geq \frac{\partial\beta}{\partial x_1}\min_{\mathcal{Q}_0}\left\{\tilde{v}^T\left(\frac{\partial^2 x_1}{\partial q^2} + \frac{\partial^2 x_2}{\partial q^2}\right)\tilde{v}\right\} \quad .$$

Rewrite $x_1 = d = \min\|q_{ij}\| = \sqrt{\min\{{q_{ij}}^T q_{ij}\}}$ and name the relative vector q_{ij} with the minimum norm, w for convenience. Since d is assumed to be differentiable, around the critical point it will hold:

$$\begin{aligned}\frac{\partial^2 x_1}{\partial q^2} &= \frac{\partial^2\left(\sqrt{\|w\|^2}\right)}{\partial^2 q^2} = \frac{\partial}{\partial q}\left(\frac{[0\cdots 0\ w\ 0\cdots 0]^T}{\|w\|}\right) \\ &= \frac{1}{\|w\|}\mathrm{diag}\left\{0,\ldots,0,\frac{\|w\|^2 I_n - ww^T}{\|w\|^2},0,\ldots,0\right\} \quad .\end{aligned}$$

On the other hand, $\frac{\partial^2 x_2}{\partial q^2} = 2\,\mathrm{diag}\{0,\ldots,0,I_n,0,\ldots,0\}$. Putting it together,

$$\begin{aligned}\min_{\mathcal{Q}_0}\tilde{v}^T\frac{\partial^2\beta}{\partial q^2}\tilde{v} \geq \\ \frac{\frac{\partial\beta}{\partial x_1}\min_{\mathcal{Q}_0}\left\{2\|w\| + 1 - \tilde{v}^T\frac{ww^T}{\|w\|^2}\tilde{v}\right\}}{\|w\|} = 2\frac{\partial\beta}{\partial x_1}.\end{aligned}$$

With reference to (26), a sufficient condition for the quadratic form $\tilde{v}^T\nabla^2\varphi\tilde{v}$ to be negative in $\mathcal{Q}_0$ is that

$$\max_{\mathcal{Q}_0}\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|} < \min_{\mathcal{Q}_0}\left\{\frac{\partial\beta}{\partial d}\right\} \quad ,$$

which is satisfied if the formation specification is valid.

Let us now consider the case where at the critical point q, $d(q)$ is not differentiable. What is interesting in this case, is that the statement of Proposition 5 can be proved without involving the Hessian. To this end, we first present a proposition from [20, cf. Proposition 3], which we relaxed slightly and therefore offer a proof.

Proposition 9: Let $V : \mathcal{Q} \to \mathbb{R}; q \mapsto \max_i f_i(q)$ be a mapping where all f_i are smooth functions. If $0 \in \mathrm{int}\,(\partial V(y))$, then y is a local minimum of V.

Proof: If f_i are smooth, then they are regular and from the pointwise maxima Theorem [18], and $\partial V(q) = \mathrm{co}\{\lim_{q_k\to q}\nabla f_i(q_k) : i \in I(q)\}$, where $I(q)$ is the set of indices for which $f_i(q) = V(q)$. Based on Lemma 2, for the origin to belong in the interior of $\partial V(q)$, it is necessary that there exists an i such that $\langle\nabla f_i(q), w\rangle > 0$, for every $w \in T_q\mathcal{Q}$ (the tangent space of $\mathcal{Q}$ at q). Following the same reasoning as in the proof of [20, Proposition 3], there must be a function f_i that increases along any direction w from point q. This implies that $V(q)$ is a local minimum. ∎

The proofs of the following two statements are similar to how the proposition above was established in [20, cf. Proposition 3]. The statements are variations of [20, Proposition 4] and [20, Proposition 5] respectively, for the case of a pointwise maximum function.

Proposition 10: Let $V(q) = \max_i f_i(q)$ where all f_i are smooth functions. At a saddle point, V is nonsmooth, and the origin is contained in $\delta(\partial V)$.

Proposition 11: Let $V(q) = \max_i f_i(q)$ where all f_i are smooth functions. At a local maximum of V, $0 = \partial V$.

In view of the above statements, the reason why nonsmooth critical points of φ can only be saddles is because the interior of a convex hull of $k \leq n$ vectors in an n-dimensional vector space is empty; all points are on the boundary. In the case of $d(q)$, even when every q_{ij} is one-dimensional, the number of any nontrivial different distances between agents cannot be more than the dimension of $\mathcal{Q}$; in other words, and with reference to (12), $|\{(i,j) : i,j \in \{1,\ldots,N\}, i \neq j\}| \leq \dim\mathcal{Q}$. Therefore, at all nonsmooth critical points any point in ∂d is a boundary point. According to [18, Theorem 2.3.9(ii)], and based on (18),

$$\partial\varphi = \overline{\mathrm{co}}\left\{\frac{\kappa\beta{\gamma_d}^{\kappa-1}\nabla\gamma_d - {\gamma_d}^{\kappa}\frac{\partial\beta}{\partial d}\zeta}{\beta^2} : \zeta \in \partial d(q)\right\}$$

which means that $\partial\varphi$ is essentially ∂d, scaled by $-\frac{{\gamma_d}^{\kappa}}{\beta^2}\frac{\partial\beta}{\partial d}$ and translated by $\frac{\kappa{\gamma_d}^{\kappa-1}}{\beta}\nabla\gamma_d$. Since all points of $\partial\varphi$ are boundary points, Proposition 10 implies that *nonsmooth* critical points of φ are necessarily saddles.

E. Proof of Proposition 6: smooth critical points are non-degenerate

Without loss of generality, assume $q \in \mathcal{B}(\epsilon)$ where $\mathcal{B}(\epsilon) = \{q : 0 < \beta < \epsilon\}$. Define $\mathcal{P}_q = \mathrm{span}\{\nabla\beta(q)\}$ and let $\mathcal{N}_q$ be the orthogonal component of $\mathcal{P}_q$ in T_q. In Proposition 5 it was shown $\xi_q|_{\mathcal{N}_q}$ is negative definite as long as $\epsilon < \epsilon_0$; the goal now is to show that $\xi_q|_{\mathcal{P}_q}$ is positive definite.

Let $v = \frac{\nabla\beta}{\|\nabla\beta\|}$. Recalling (22), at a critical point we have $\kappa\beta\nabla\gamma_d = \gamma_d\nabla\beta$. Similarly to the proof of Proposition 5, taking outer products in (22) yields (23), and substituting in (21) we get (24) as an expression of the Hessian at a critical point. Here, we also take squared norms on both sides of (22) and solve for $\kappa\beta$

$$(\kappa\beta)^2\|\nabla\gamma_d\|^2 = \gamma_d^2\|\nabla\beta\|^2 \Rightarrow \kappa\beta = \frac{\gamma_d}{4\kappa\beta}\|\nabla\beta\|^2 \quad , \quad (28)$$

exploiting the fact that $\|\nabla\gamma_d\|^2 = 4\gamma_d$, and then substitute for $\kappa\beta$ from (28) into (21), to get

$$\nabla^2\varphi(q) = \frac{{\gamma_d}^{\kappa}}{\beta^2}\left[\frac{\|\nabla\beta\|^2}{2\kappa\beta}I + \left(1-\frac{1}{\kappa}\right)\frac{1}{\beta}\nabla\beta\,\nabla\beta^T - \nabla^2\beta\right] \ .$$

Evaluating $\xi_q|_{\mathcal{P}_q}$ with $\upsilon = \frac{\nabla\beta}{\|\nabla\beta\|}$ yields

$$\begin{aligned}&\upsilon^T\,\nabla^2\varphi\,\upsilon = \\ &\frac{{\gamma_d}^{\kappa}}{\beta^2}\left[\frac{\|\nabla\beta\|^2}{2\kappa\beta} + \left(1-\frac{1}{\kappa}\right)\frac{1}{\beta}\left(\frac{\nabla\beta^T\nabla\beta}{\|\nabla\beta\|}\right)^2 - \upsilon^T\nabla^2\beta\,\upsilon\right] \\ &\qquad = \frac{{\gamma_d}^{\kappa}}{\beta^2}\left[\frac{2\kappa-1}{2\kappa\beta}\|\nabla\beta\|^2 - \upsilon^T\nabla^2\beta\,\upsilon\right] \ .\end{aligned}$$

Thus a sufficient condition for $\upsilon^T\nabla^2\varphi\,\upsilon$ to be positive is

$$\min_{\mathcal{Q}_0}\left\{\frac{2\kappa-1}{2\kappa\beta}\|\nabla\beta\|^2\right\} - \max_{\mathcal{Q}_0}\left\{\upsilon^T\nabla^2\beta\upsilon\right\} > 0 \qquad (29)$$

Expanding $\nabla^2\beta$ similarly to the proof of Proposition 5,

$$\begin{aligned}\max_{\mathcal{Q}_0}\left\{\upsilon^T\nabla^2\beta\upsilon\right\} &\le \frac{\partial^2\beta}{\partial {x_1}^2}\max_{\mathcal{Q}_0}\left\{\upsilon^T\left[\frac{\partial x_1}{\partial q}\left(\frac{\partial x_1}{\partial q}\right)^T\right]\upsilon\right\} \\ &\quad + \frac{\partial\beta}{\partial x_1}\max_{\mathcal{Q}_0}\left\{\upsilon^T\left[\frac{\partial^2 x_1}{\partial q^2}\right]\upsilon\right\} \\ &\quad + \frac{\partial^2\beta}{\partial {x_2}^2}\max_{\mathcal{Q}_0}\left\{\upsilon^T\left[\frac{\partial x_2}{\partial q}\left(\frac{\partial x_2}{\partial q}\right)^T\right]\upsilon\right\} \\ &\quad + \frac{\partial\beta}{\partial x_2}\max_{\mathcal{Q}_0}\left\{\upsilon^T\left[\frac{\partial^2 x_2}{\partial q^2}\right]\upsilon\right\} \\ &\le \frac{\partial^2\beta}{\partial {x_1}^2}(1+4\|w\|^2) + \frac{\partial\beta}{\partial x_1}\left(\frac{2\|w\|+1}{\|w\|}\right) \ , \qquad (30)\end{aligned}$$

simplified by the fact that $\frac{\partial\beta}{\partial x_1} = \frac{\partial\beta}{\partial x_2}$ and $\frac{\partial^2\beta}{\partial {x_1}^2} = \frac{\partial^2\beta}{\partial {x_2}^2}$. The terms $\frac{\partial^2\beta}{\partial {x_1}^2}$ and $\frac{\partial\beta}{\partial x_1}$ that appear in the right hand side of (30), are both positive and bounded when $q \in \mathcal{B}(\epsilon)$, with bounds dependent on parameters ζ and d_0 (see (8)). Indeed, for a sufficiently small ϵ, $\frac{\partial^2\beta}{\partial {x_1}^2}$ is decreasing in $\mathcal{B}(\epsilon)$ and can be upper bounded by its limit as $x_1 \to d_0$, whereas $\frac{\partial\beta}{\partial x_1}$ is increasing and can be upper bounded by its value at $d = \beta^{-1}(\epsilon)$. The corresponding bounds are

$$\begin{aligned}\max_{\mathcal{Q}_0}\frac{\partial\beta}{\partial x_1} &\le 2(\mu e^{-\epsilon}-1)\sqrt{\log\left(\frac{a}{\mu - e^{\epsilon}}\right)} \\ \max_{\mathcal{Q}_0}\frac{\partial^2\beta}{\partial {x_1}^2} &\le \frac{2a[\mu(1-2(d_0^2+d_0-r)^2)e^{(d_0^2+d_0-r)^2}+a]}{(a-\mu e^{(d_0^2+d_0-r)^2})^2}.\end{aligned}$$

Note that the former increases almost linearly with ζ (see (8), and in particular how a and r depend on ζ), while the latter decreases with ζ. On the other hand,

$$\begin{aligned}\min_{\mathcal{Q}_0}\left\{\frac{\|\nabla\beta\|^2}{\beta}\right\} &= \min_{\mathcal{Q}_0}\left\{\frac{1}{\beta}\left(\frac{\partial\beta}{\partial x_1}\right)^2\left(2w+\frac{w}{\|w\|}\right)^2\right\} \\ &\ge (2d_0+1)^2\min_{\mathcal{Q}_0}\left\{\frac{1}{\beta}\left(\frac{\partial\beta}{\partial x_1}\right)^2\right\} \\ &= \frac{(2d_0+1)^2}{\epsilon}\min_{\mathcal{Q}_0}\left\{\left(\frac{\partial\beta}{\partial x_1}\right)^2\right\} \ ,\end{aligned}$$

since $\max_{\mathcal{Q}_0}\beta = \epsilon$. Note that $\frac{\partial\beta}{\partial x_1}$ is lower bounded in $\mathcal{Q}_0$, with its value at $d = d_0$ being equal to $\frac{\zeta}{2d_0+1}$. Thus we have the first term of (29) rising quadratically with ζ, while the second increasing at most linearly in $\mathcal{Q}_0$. Therefore, for sufficiently large ζ and small ϵ, (29) can be satisfied.

F. Proof of Proposition 7: nonsmooth critical points are non-degenerate

Assume that q is a nonsmooth critical point of φ. According to Definition 5, we have

$$\sum_{i\in\hat{I}}\lambda_i\,\mathrm{d}\varphi_i = \sum_{i\in\hat{I}}\lambda_i\frac{\beta\,\mathrm{d}\gamma - \gamma\,\mathrm{d}\beta_i}{\beta^2} = 0 \ , \qquad (31)$$

where we dropped the subscript to β because all the $\beta_i = \beta_j$ for all $i,j \in \hat{I}(q)$. Substituting for γ, (31) implies $\kappa\beta\,\mathrm{d}\gamma_d - \gamma_d\sum_i\lambda_i\,\mathrm{d}\beta_i = 0$ which after taking inner product with $\mathrm{d}\gamma_d$ yields $\kappa\beta\,\|\mathrm{d}\gamma_d\|^2 - \gamma_d\left(\sum_{i\in\hat{I}}\lambda_i\,\mathrm{d}\beta_i\right)^T\mathrm{d}\gamma_d = 0$. Since$\|\mathrm{d}\gamma_d\|^2 = 4\gamma_d$ we see that at q,

$$0 < \kappa\beta = \frac{1}{4}\,\mathrm{d}{\gamma_d}^T\sum_{i\in\hat{I}}\lambda_i\,\mathrm{d}\beta_i \ . \qquad (32)$$

Now, consider a vector υ in the tangent space $\hat{T}(q)$. According to Definition 7, $\hat{T}(q) = \cap_{i\in\hat{I}(q)}\mathrm{kern}\,\mathrm{d}\varphi_i(q)$ suggesting that $\upsilon \in \cap_{i\in\hat{I}}\mathrm{kern}\,\mathrm{d}\varphi_i = \cap_{i\in\hat{I}}\mathrm{kern}\{\beta\,\mathrm{d}\gamma - \gamma\,\mathrm{d}\beta_i\}$. Then, as υ is orthogonal to $\beta\,\mathrm{d}\gamma - \gamma\,\mathrm{d}\beta_i$, $\forall i \in \hat{I}$, we have $\upsilon^T(\beta\kappa\gamma_d^{\kappa-1}\,\mathrm{d}\gamma_d - \gamma_d^{\kappa}\,\mathrm{d}\beta_i) = 0$, which gives

$$\upsilon^T\mathrm{d}\gamma_d = \frac{\gamma_d}{\kappa\beta}\mathrm{d}{\beta_i}^T\upsilon \qquad (33)$$

Recalling again that υ is orthogonal to $\mathrm{d}\varphi_i$, we have

$$\begin{aligned}\upsilon^T\left(\sum_{i\in\hat{I}}\lambda_i\,\mathrm{d}^2\varphi_i\right)\upsilon &= \upsilon^T\left(\sum_{i\in\hat{I}}\lambda_i\frac{\beta\,\mathrm{d}^2\gamma - \gamma\,\mathrm{d}^2\beta_i}{\beta^2}\right)\upsilon \\ &= \upsilon^T\left(\frac{\mathrm{d}^2\gamma}{\beta}\sum_{i\in\hat{I}}\lambda_i\right)\upsilon - \upsilon^T\left(\frac{\gamma}{\beta^2}\sum_{i\in\hat{I}}\lambda_i\,\mathrm{d}^2\beta_i\right)\upsilon \\ &= \frac{\kappa\gamma_d^{\kappa-2}}{\beta}\upsilon^T\left\{(\kappa-1)\left[\mathrm{d}\gamma_d\,\mathrm{d}\gamma_d^T\right] + 2\gamma_d\,I\right\}\upsilon \\ &\quad - \frac{\gamma_d^{\kappa}}{\beta^2}\upsilon^T\left(\sum_{i\in\hat{I}}\lambda_i\mathrm{d}^2\beta_i\right)\upsilon \\ &= \frac{(\kappa-1)\kappa\beta\|\upsilon^T\mathrm{d}\gamma_d\|^2 + 2\kappa\beta\gamma_d - \gamma_d^2\upsilon^T\sum\lambda_i\,\mathrm{d}^2\beta_i\,\upsilon}{\gamma_d^{2-\kappa}\beta^2}.\end{aligned} \qquad (34)$$

Two remarks are in order here: first, due to (33), it is $\|\upsilon^T\mathrm{d}\gamma_d\|^2 = \frac{\gamma_d^2}{\kappa^2\beta^2}\|\mathrm{d}{\beta_i}^T\upsilon\|^2$ for all i; second, due to (32), and although in general increasing κ decreases β—due to ϵ—their product is upper bounded in $\mathcal{Q}_0$. To take into account these dependencies, we use (32) and (33) and rewrite (34) as

$$v^T \sum_{i\in\hat{I}} \lambda_i \, \mathrm{d}^2\varphi_i \, v = \frac{\left(\frac{\mathrm{d}\gamma_d^T \sum_{i\in\hat{I}} \lambda_i \, \mathrm{d}\beta_i}{0.5\gamma_d} + \frac{(\kappa-1)\left\|v^T \mathrm{d}\beta_i\right\|^2}{\mathrm{d}\gamma_d^T \sum_{i\in\hat{I}} \lambda_i \, \mathrm{d}\beta_i}\right) I - \sum_{i\in\hat{I}} v^T \lambda_i \, \mathrm{d}^2\beta_i v}{\gamma_d^{-\kappa}\beta^2} \tag{35}$$

which is more involved, only so that we can consider the following two cases: a) $v \not\perp \mathrm{d}\beta_i$, and b) $v \perp \mathrm{d}\beta_i$, which both have to hold for all $i \in \hat{I}(q)$ due to (33).

When $v \not\perp \mathrm{d}\beta_i$, $\forall i \in \hat{I}(q)$, note first that due to (33), the first term in the numerator of the right hand side of (35) is strictly positive, and that κ can be increased sufficiently high to enforce the positive sign of $\mathrm{d}\gamma_d^T \sum_i \lambda_i \, \mathrm{d}\beta_i$ to the whole expression in the numerator.

When $v \perp \mathrm{d}\beta_i$, $\forall i \in \hat{I}(q)$, then the second term in the parenthesis in the numerator at the right hand side of (35) vanishes, and (35) reduces to

$$\begin{aligned} v^T \sum_{i\in\hat{I}} \lambda_i \, \mathrm{d}^2\varphi_i \, v &= \frac{\frac{\mathrm{d}\gamma_d^T \sum_{i\in\hat{I}} \lambda_i \, \mathrm{d}\beta_i}{0.5\gamma_d} I - v^T \sum_{i\in\hat{I}} \lambda_i \, \mathrm{d}^2\beta_i \, v}{\gamma_d^{-\kappa}\beta^2} \\ &= \frac{\gamma_d{}^{\kappa}}{\beta^2}\left\{2\left(\frac{\mathrm{d}\gamma_d}{\|\mathrm{d}\gamma_d\|}\right)^T \frac{\sum \lambda_i \, \mathrm{d}\beta_i}{\|\mathrm{d}\gamma_d\|} - v^T \sum \lambda_i \, \mathrm{d}^2\beta_i \, v\right\} \\ &\le \frac{\gamma_d{}^{\kappa}}{\beta^2}\left\{2 \max_{\mathcal{Q}_0} \frac{\|\sum \lambda_i \, \mathrm{d}\beta_i\|}{\|\mathrm{d}\gamma_d\|} - \min_{\mathcal{Q}_0} \lambda\big(\sum \lambda_i \, \mathrm{d}^2\beta_i\big)\right\} \quad , \end{aligned} \tag{36}$$

where $\lambda(\cdot)$ denotes an eigenvalue. Obviously, if the right hand side of (36) is negative, the left hand side has to be also. Relating $\sum \lambda_i \mathrm{d}\beta_i$ to $\nabla\beta$ and $\sum \lambda_i \, \mathrm{d}^2\beta_i$ to $\nabla^2\beta$, the similarity between (36) and (25) becomes apparent, and thus choosing ζ appropriately large along the lines of the proof of Proposition 5, makes $v^T \sum_{i\in\hat{I}} \lambda_i \, \mathrm{d}^2\varphi_i \, v$ negative definite.

Thus, (35) can be made non-singular—either positive or negative definite—by a proper selection of ζ.

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