Modeling of Multiple Mobile Manipulators Handling a Common Deformable Object

Herbert G. Tanner,
Kostas J. Kyriakopoulos,* and
N. J. Krikelis
Control Systems Laboratory
Department of Mechanical Engineering
Division of Machine Design and Control
National Technical University of Athens
Athens, 10682 Greece
e-mail: {htanner, kkyria}@entral.ntua.gr

Received January 9, 1998; accepted May 27, 1998

The dynamic equations of motion for a system of multiple mobile manipulators carrying a common deformable object are developed. The derived centralized model is based on Kane's approach to dynamics. The imposed kinematic nonholonomic constraints are included and incorporated into the dynamics. It is pointed out, however, that these kinematic constraints are only necessary, and they only become sufficient to impose nonholonomic motion when accompanied by the dynamic counterparts. Sufficient conditions for avoiding tipping over by the mechanisms are also provided. The whole set of constraint equations is analyzed and useful properties for the set of admissible solutions for the ground reaction forces are obtained. The deformable object under manipulation is modeled and the simplest approximating grid structure is indicated.

1. INTRODUCTION

Until recently, robotic manipulators were mechanisms rigidly attached to the ground. The need to perform difficult or/and dangerous operations in space generated the study and the consequent construction of mobile manipulators, i.e., robotic manipulators whose bases could move.[1,2]

Now, mobile manipulators are not meant to operate solely in space. Mobile manipulation operating in aquatic environments is a recent and active area of research.[3,4] Of course, mobile manipulators can also operate on ground, given they are attached to a vehicle (normally wheeled), and some signifi-

* To whom all correspondence should be addressed.

CCC 0741-2223/98/110599-25

cant research has already been conducted in the field.[5–11]

Mobile manipulators extend the manipulator workspace and its ability to work efficiently. Due to the mobile base, the manipulator is capable of configuring itself to practically any operational point. In addition, it can grasp and manipulate an object in many different ways (positions and orientations). This actually means that it is a redundant mechanism, with all the inherent capabilities and problems of such systems.

However, as is always the case, there is a price to pay for advantages: more difficulty in control. This difficulty is due mainly to a class of motion constraints called nonholonomic.[12–16] These constraints are equations involving the generalized coordinates and their derivatives in a way that makes them nonintegrable. Thus, the dimension of the configuration space cannot be reduced. Under the effect of such constraints the system maintains its controllability,[17] but is generally harder to steer.

Many of the models proposed for mobile manipulator systems do not include nonholonomic constraints.[3,7] These models focus on the interaction of the system with its environment. Tarn et al.[3] model an underwater vehicle with multiple robotic manipulators. Dynamic modeling is achieved by the use of Kane's dynamic equations.[18] The motion of the autonomous underwater vehicle (AUV) and momentum preservation laws impose nonholonomic constraints which are not included in the model. Khatib et al.[7] model cooperating mobile manipulators, the mobile platforms of which can move on a planar surface. The dynamic equations of motion are obtained using the classic Euler–Lagrange formulation in operational space,[19] assuming that the mobile platforms can move in a holonomic way.

Other models include nonholonomic constraints using classic Euler–Lagrange[5] and Newton–Euler formulations.[9] These models are more accurate and consistent, even though they still do not consider all constraints imposed in mobile manipulator systems. In ref. 9, the no-slipping nonholonomic condition is not taken into account nor is the angular momentum preservation condition associated with tipping over, which is also missing in the otherwise complete analysis in ref. 5. Chen and Zalzala[9] include in their model the nonholonomic constraint associated with the no-slipping condition. Inclusion of the constraint in the model is achieved by algebraic manipulation. Yamamoto et al.[5,6] go further, including the nonholonomic constraint resulting from the no-skidding condition. Merging the constraints with the system's equations of motion is done using the well known Lagrange multipliers methodology. Dubowsky and Vance[8] discuss the possibility of the mobile platform tipping over due to dynamic interaction with the attached manipulator.

Recently, Thanjavur and Rajagopala[20] modeled an autonomously guided vehicle (AGV) using Kane's equations. They pointed out the merits of using Kane's approach to model vehicles and utilized some of the tools to incorporate nonholonomy. They focused on the dynamics of the vehicle individual components (drive and castor wheels, drive wheel assemblies, etc.) and included only the classical no-slipping and no-skidding constraints, and did not provide any necessary conditions for these constraints to be satisfied.

In this article, a model within the framework of Kane's approach is proposed. As opposed to Euler–Lagrange and Newton–Euler formulations, Kane's methodology involves less arithmetic operations and is thus simpler, faster in simulation, and requires less effort in construction.[3,21] Kane's equations can be easily brought to a closed form,[3] which is best suited for control purposes.

The multiple mobile manipulators involved interact through the common deformable object (Fig. 1). The object is arbitrarily shaped and deformable. In such cases, the methods developed for object handling and manipulation by multiple manipulators cannot be applied, since the object has infinite degrees of freedom that do not restrict one rigid grasp relative to another. Moreover, the dynamics of the body cannot be described in a straightforward manner, since its mass center and every other point can move relative to the fixed grasp points.

The deformable object is modeled by its elastodynamic equations. Then the notion of operational space is expanded to include more dimensions than a common task space, yielding a generalized operational space. Within this new framework, the simplest approximating grid structure for the deformable object becomes evident. This simplification, along with the generalized operational point concept, enables the application of Khatib's[22] augmented object approach to merge the models developed for each mobile manipulator with the object model and produce a compact set of dynamic equations for the system.

The combined system of the mobile manipulators linked by the manipulated object does not operate in space. The mobile platforms on which the

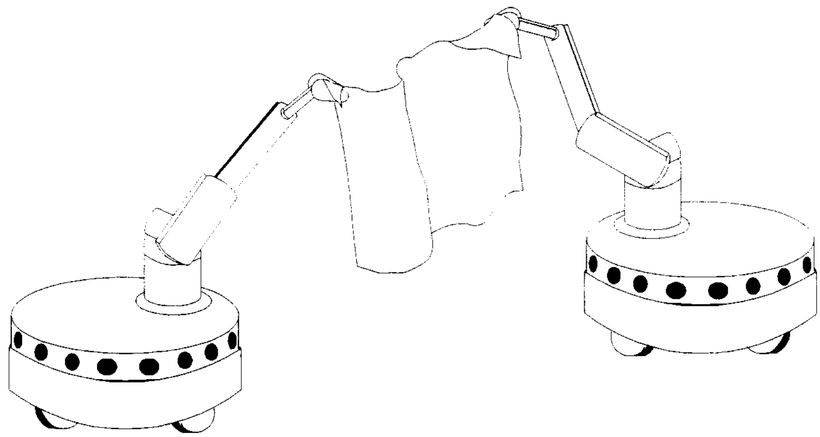

Figure 1. A multiple mobile manipulation system.

manipulators are based move on the ground and thus interact with each other in the same way as the legs of a walking robot do, forming a closed chain. This closed chain can be visualized if one models the platforms as an unactuated joint virtual manipulator connecting each wheel to the ground.[23]

A systematic approach to the analysis of closed-chain systems has been presented by Tarn et al.[24] They constructed a model for two cooperating manipulators calculating the Lagrange equations analytically. The process for obtaining the Lagrange function is driven by the peculiarity of the mechanism[25] and the complexity of the involved calculations. Another difficulty is finding recursive computational schemes. Moreover, the computing time is relatively long for real-time controllers.[26] Lin[27] also reported difficulties arising from obscure dynamic phenomena. A different, popular approach to the analysis of closed-link mechanisms, originally introduced by Smith,[28] is to cut the closed-chain mechanism at several joints, transforming it to an open chain. This approach was extended by other researchers.[29] A milestone in the study of closed-chain mechanism is the work of Luh and Zheng,[26] who replaced the problem of computing the dynamics of closed-chain spatial systems to that of open-link tree-structure mechanisms subjected to unknown joint torques at the cut points. In their work, they made use of d'Alembert's principle and employed Lagrange multipliers to compute the unknown torques.

However, computation of Lagrange multipliers is known to be a cumbersome and computationally inefficient task. Nakamura and Ghodoussi[25] proposed a general computational scheme of the inverse dynamics of the closed-link mechanisms which is also based on d'Alembert's principle but without computing the Lagrange multipliers. The scheme is computationally efficient compared to conventional methods, however the choice for the cut joints of the closed kinematic loops depends on the redundancy of the actuators.[30]

Our approach to the closed chain formed follows the same philosophy. The closed chain is cut at the points of contact between the wheels of each vehicle and the ground. However, we do not proceed to calculate the forces exerted there, since it is well known that they cannot be resolved.[8] Instead, we define the space of admissible values for them through a set of constraint equations. Among the duties of the control scheme that could be applied is ensuring that these forces stay within specified limits. This is accomplished by controlling the kinematic quantities of each mechanism in a way indicated by the constraint equations.

The most significant constraints imposed on the system are formally stated in equation form. It is pointed out that conventional velocity constraint equations associated with no slipping and no skidding are only necessary conditions for nonholonomic motion. By themselves, they cannot guarantee nonholonomic motion; they only describe the result. The cause of nonholonomic motion is the ground reaction forces at the wheels. We state additional dynamic constraints involving the ground reaction forces which eventually provide a sufficient set of conditions for nonholonomy. The requirement for avoiding tipping over is also taken into account.

Conditions for avoiding tipping over and maintaining static friction are also stated in ref. 8, where the objective is to maintain the vehicle stationary and constraints emerge from the contact conditions of the vehicle's outriggers with the ground. This could not be applied in our case, however, because now tipping over can also be caused by the combined effect of the vehicle motion. We need to formulate a condition for which the static conditions of ref. 8 will be a special case.

Modeling mobile manipulation systems has not received much attention yet, with few exceptions.[7,31] We hope to contribute in this direction with the main results of this article, which can be summarized as follows:

- The model is constructed with the use of Kane's dynamical equation, which possesses a number of merits compared to Euler–Lagrange and Newton–Euler formulatinos in terms of both simplicity and computational effort.
- Object manipulation is not restricted to rigid materials.
- Deformable object modeling does not neglect any of the object dynamics, using three-dimensional elastodynamic equations.
- Introducing the concept of generalized operational space, it is possible to apply proposed task space methodologies in the case of infinite degrees of freedom deformable objects.
- The whole system is regarded as a closed-chain mechanism.
- Nonholonomic constraints imposed on the mobile bases are taken into account.
- Conventional nonholonomic constraint equations, which are only necessary conditions for nonholonomy, are accompanied by constraint equations at the dynamic level which complete the set and make it sufficient.
- Additional sufficient conditions regarding the system's mechanical stability are included.

Analytical expressions for the equations of motion are not given, since there is no specification on the number of manipulator links, thereby to account for both nonredundant and redundant manipulators. For the vehicle and the first manipulator link, calculations are carried out analytically to present our approach to nonholonomic issues with clarity and to reveal the nature of the interaction between the manipulator and the vehicle. The remainder of the procedure for the manipulator is described and specific references are given.

The rest of the article is organized as follows. In section 2, Kane's methodology is applied to the problem of modeling a single mobile manipulator. Section 3 is devoted to modeling the deformable object. The combined model for the system of multiple mobile manipulators handling a common deformable object is constructed in section 4. In section 5, the dynamic constraints imposed on the system are presented and analyzed. Section 6 summarizes conclusions drawn from the present work.

2. MODELING OF A MOBILE MANIPULATOR

2.1. Preliminaries

This section is devoted to modeling one mobile manipulator. Modeling the rest of the set of mobile manipulators follows the same lines. Let this mobile manipulator be denoted by k. Consider a fixed inertial frame, $\{I\}$ as in Figure 2. Its $\mathbf{x}$ and $\mathbf{y}$ axes lie on the horizontal plane. On the mobile platform of mobile manipulator k, a frame $\{v_k\}$ is attached at its mass center. The $\mathbf{x}_k^v$ axis is aligned with the vehicle linear velocity and $\mathbf{z}_k^v$ is parallel to $\mathbf{z}^I$. At the mass center, the center principal inertial frame, $\{v_{\mathrm{cpi}_k}\}$, is assigned, which can generally differ from $\{v_k\}$. The mounting point of the attached manipulator is named mm_k and coincides with the platform point $\overline{mm}_k$ (the distinction between the two points will be justified in the sequel). On the serial link manipulator, the frames are assigned according to the Denavit–Hartenberg convention. Finally, a coordinate frame is assigned to a fixed point of the manipulator end effector which is in continuous contact with the deformable object under manipulation. In the sequel, the superscript on the right side of the symbol will denote the rigid body to which the quantity refers. The superscript on the left will denote the reference frame with respect to which the quantity is expressed. When omitted, frame $\{I\}$ is assumed. Since this whole section refers to mobile manipulator k, the subscript k will be dropped for simplicity.

Frame $\{w_j\}$ is attached at the center of each wheel j, $j = 1, \ldots, w$, of the vehicle. Its $\mathbf{z}^{w_j}$ axis remains parallel to the vehicle $\mathbf{z}^v$ axis, but the frame can rotate around this axis. Each frame is related to the fixed frame through a rotation matrix $\mathbf{R}$. This

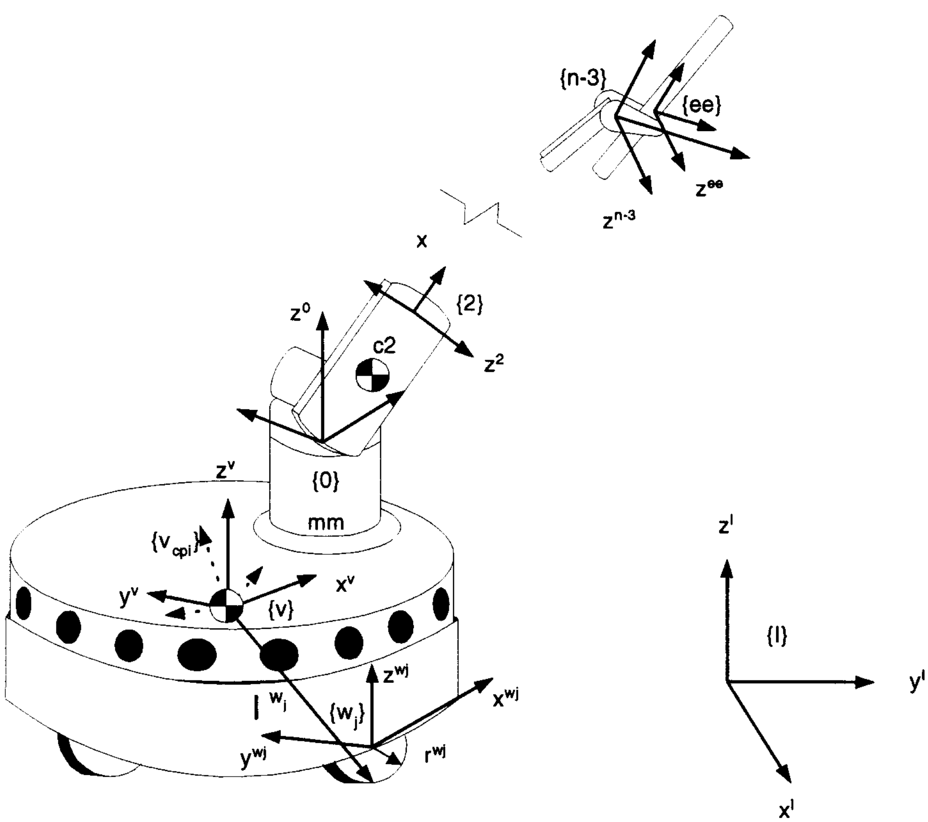

Figure 2. Frame assignment on mobile manipulator k.

rotation matrix will be written as, e.g., ${}^{I}_{w_j}\mathbf{R}$, to indicate transformation of free vectors, expressed in frame $\{w_j\}$, to the inertial frame. It should also be noted that bound vectors can be freely transformed from frame to frame via the rotation matrix. Thus, for frames $\{A\}$ and $\{B\}$ with coincident origins, where $\{B\}$ is rotating relative to $\{A\}$, the position vector of a fixed point P in $\{B\}$, ${}^{B}\mathbf{p}$, is the same with ${}^{A}\mathbf{p}$, although the local coordinate representations differ. These local representations are related through the rotation matrix. This is not the case with velocities; while ${}^{B}\mathbf{v}^{P}=0$, ${}^{A}\mathbf{v}^{P}$ does not.

In the sequel it is assumed that the motion of all vehicles is restricted on the horizontal plane. Moreover, the mass of the vehicle wheels is neglected. As a result, no inertial forces or torques are exerted on the wheels. This assumption is valid, since the wheels' mass is generally negligible compared to the vehicle mass. The coefficient of static friction between each wheel and the ground is assumed to be the same for all wheels and equal to μ.

The attached manipulator is supposed to have $n-3$ rotational joints. The issue of redundancy will not be addressed in this paper. The assumption that the joints are rotational can be easily relieved; however, since the mixed use of both prismatic and rotational joints will complicate some analytical derivations, the authors feel they should preserve the clarity of presentation by using one type only.

If there are n degrees of freedom in the system, then let the generalized coordinates of the system be

$$\mathbf{q}=[\,x^{v} \quad y^{v} \quad \theta \quad q^{1} \quad \cdots \quad q^{n-3}]$$

where x and y are the vehicle planar coordinates in the fixed coordinate system $\{I\}$, θ is the angular velocity about axis $\mathbf{z}^{I}$, and the qs correspond to the manipulator joints. The number of generalized coordinates remains n, since the constraints imposed on the platform are completely nonholonomic and do not restrict the motion of the system to a constrained configuration submanifold, as can be verified by the controllability rank condition.[17]

The generalized speeds are defined as

$$\mathbf{u}=\left[\,\dot{x}^{v} \quad \dot{y}^{v} \quad \dot{\theta} \quad \dot{q}^{1} \quad \cdots \quad \dot{q}^{n-3} \quad \cdots\right]$$

Normally, when analyzing a specific mechanism, generalized speeds are defined at a later stage. Choosing generalized speeds has many advantages, one of them being the simplification of the dynamic equations. This is done by appropriately defining the generalized speeds after inspection of the resulting equations, so that expressions can take the simplest form possible. This advantage provided by Kane's approach will be spared here for the sake of generality of analysis. A second advantage of using generalized speeds is the ability to bring forces that do not contribute to the dynamic equations into evidence.[18] This is the reason for not specifying the exact number of generalized speeds at this stage.

2.2. Analysis of the Vehicle

2.2.1. The Vehicle Wheels

A wheel can have two degrees of freedom. One permits it to rotate about the $\mathbf{y}^{w_j}$ axis. This degree of freedom is controlled by an input torque τ^{w_j}. We believe that when controlling a vehicle that can be modeled as a unicycle,[16] one does not need to know the rolling angle of the wheel. The rate of change of this angle can be extracted by the nonholonomic constraints associated with rolling without slipping. The respective angular acceleration is not needed since we have neglected the wheel mass. For the above reasons, we have not included this degree of freedom in the set of generalized coordinates.

The second degree of freedom corresponds to the steering angle of the wheel and is uniquely determined by the vehicle orientation angle through the cart steering system equations. Therefore, it cannot be an additional degree of freedom. Equivalently, it can be considered as an additional degree of freedom, but then it must be eliminated through the holonomic equations relating it with the vehicle orientation angle. Let $\phi^{w_j}(\theta)$ be the wheel steering angle relative to the vehicle frame, caused by the steering mechanism. The dependence of ϕ^{w_j} on θ will be dropped in the sequel.

The following contact forces (and torques) are identified on the wheel.

1. The reaction from the ground $\mathbf{F}^{w_j}$, applied at the point of contact between the wheel and the ground.
2. The reaction from the vehicle body $\mathbf{F}^{v_j}$.
3. The control input torques $\mathbf{T}^{w_j}$, both applied at the frame's $\{w_j\}$ origin.

As stated in ref. 18, the *ground reactions* to the wheels, as well as the vehicle–wheel *interaction forces*, do not contribute to the dynamic equations. In the first case, the point of contact between the wheel and the ground has zero instantaneous velocity which results in a null contribution, since in Kane's equation each force is dot-multiplied by a partial derivative of the point's velocity. In the latter case, the velocity of each rigid body is the same, whereas the forces are opposite by law of action and reaction. When added, these products are eliminated. This leaves the *input torque* as the only contributing external influence.

The wheels are subject to nonholonomic constraints which can be merged into the wheel equations of motion. This inclusion results in a smaller number of dynamic equations. Thus, the system representation is simplified with the nonholonomic constraint equations taking the place of the missing dynamic equations. The velocities of the wheels are directly related to the nonholonomic constraints, and this fact will be used to derive expressions for them.

The velocity in $\{I\}$ of the frame's $\{w_j\}$ origin can be expressed as a function of the generalized speeds

$$\mathbf{v}^{w_j} = u_1 \mathbf{x}^I + u_2 \mathbf{y}^I + u_3(\mathbf{z}^v \times \text{ell}^{w_j})$$

where ell^{w_j} is the position vector from the origin of frame $\{v\}$ to the contact point between the wheel and the ground (Fig. 2). The expression of this velocity in frame $\{w_j\}$ is calculated as

$$\mathbf{v}^{w_j} = u_1\, {}^{w_j}_{I}\mathbf{R}\mathbf{x}_I + u_2\, {}^{w_j}_{I}\mathbf{R}\mathbf{y}_I + u_3\, {}^{w_j}_{v}\mathbf{R}({}^{v}\mathbf{z}_v \times {}^{v}\text{ell}^{w_j})$$

The rotation matrices ${}^{w_j}_{I}\mathbf{R}$ and ${}^{w_j}_{v}\mathbf{R}$ can be expressed as

$${}^{w_j}_{I}\mathbf{R} = \begin{bmatrix} \cos(\phi^{w_j}+\theta) & \sin(\phi^{w_j}+\theta) & 0 \\ -\sin(\phi^{w_j}+\theta) & \cos(\phi^{w_j}+\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{w_j}_{v}\mathbf{R} = \begin{bmatrix} \cos\phi^{w_j} & \sin\phi^{w_j} & 0 \\ -\sin\phi^{w_j} & \cos\phi^{w_j} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Utilizing the above expressions, $\mathbf{v}^{w_j}$ can be rewritten

$$\begin{aligned}\mathbf{v}^{w_j} = \Big[& u_1 \cos(\phi^{w_j}+\theta) + u_2 \sin(\phi^{w_j}+\theta) \\ & + u_3\big({}^{v}l_x^{w_j} \sin\phi^{w_j} - {}^{v}l_y^{w_j} \cos\phi^{w_j}\big)\Big]\mathbf{x}^{w_j} \\ & + \Big[u_2 \cos(\phi^{w_j}+\theta) - u_1 \sin(\phi^{w_j}+\theta) \\ & + u_3\big({}^{v}l_y^{w_j} \sin\phi^{w_j} + {}^{v}l_x^{w_j} \cos\phi^{w_j}\big)\Big]\mathbf{y}^{w_j}\end{aligned}$$

The *no-skidding* condition implies that the second term vanishes:

$$0 = u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta) + u_3\left({}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}\right)$$

$$\Rightarrow \quad u_3 = \frac{u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \tag{2.1}$$

Therefore, $\mathbf{v}^{w_j}$ can be written

$$\mathbf{v}^{w_j} = \left[u_1 \cos(\phi^{w_j} + \theta) + u_2 \sin(\phi^{w_j} + \theta) + u_3\left({}^v l_x^{w_j} \sin \phi^{w_j} - {}^v l_y^{w_j} \cos \phi^{w_j}\right)\right]\mathbf{x}^{w_j} \tag{2.2}$$

The *no-slipping* constraint can be expressed as

$$\mathbf{v}^{w_j} = \boldsymbol{\omega}^{w_j} \times r^{w_j}\mathbf{z}^{w_j} \tag{2.3}$$

where r^{w_j} is the radius of the wheel.

To express this nonholonomic constraint in terms of generalized speeds, we have to introduce an additional generalized speed, u_{w_j}, which will be eliminated presently. This generalized speed represents the angular velocity measure of the wheel around its $\mathbf{y}^{w_j}$ axis:

$$\boldsymbol{\omega}^{w_j} = \boldsymbol{\omega}^{v} + {}^v\boldsymbol{\omega}^{w_j} \overset{\Delta}{=} \boldsymbol{\omega}^{v} + u_{w_j}\mathbf{y}^{w_j} + u_3 \frac{\partial \phi^{w_j}}{\partial \theta}\mathbf{z}^{w_j} \tag{2.4}$$

Substituting (2.2) and (2.4) into Eq. (2.3), and after some algebraic manipulation, yields

$$u_{w_j} = \frac{1}{r^{w_j}}\left[u_1 \cos(\phi^{w_j} + \theta) + u_2 \sin(\phi^{w_j} + \theta) + u_3\left({}^v l_x^{w_j} \sin \phi^{w_j} - {}^v l_y^{w_j} \cos \phi^{w_j}\right)\right] \tag{2.5}$$

Then $\boldsymbol{\omega}^{w_j}$ can be expressed as

$$\boldsymbol{\omega}^{w_j} = \frac{u_1 \cos(\phi^{w_j} + \theta) + u_2 \sin(\phi^{w_j} + \theta) + u_3\left({}^v l_x^{w_j} \sin \phi^{w_j} - {}^v l_y^{w_j} \cos \phi^{w_j}\right)}{r^{w_j}}\mathbf{y}^{w_j} + u_3\left(1 + \frac{\partial \phi^{w_j}}{\partial \theta}\right)\mathbf{z}^{w_j}$$

The last term is a result of the wheel rotation due to the vehicle angular motion and to the relative motion between the wheel and the vehicle caused by the steering mechanism. The generalized speed u_{w_j} has already been eliminated. After eliminating u_3 from Eq. (2.1) we obtain

$$\boldsymbol{\omega}^{w_j} = \left[\frac{u_1 \cos(\phi^{w_j} + \theta) + u_2 \sin(\phi^{w_j} + \theta)}{r^{w_j}} + \frac{\left({}^v l_x^{w_j} \sin \phi^{w_j} - {}^v l_y^{w_j} \cos \phi^{w_j}\right) \times (u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta))}{r^{w_j}\left({}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}\right)}\right]\mathbf{y}^{w_j} + \frac{(1 + (\partial \phi^{w_j}/\partial \theta)) \times (u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta))}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}}\mathbf{z}^{w_j}$$

Using Eq. (2.1) in all expressions that follow, all velocities are described using a subset of the original generalized speeds, namely, $n-1$. This will reduce the number of resulting dynamic equations, which will again be n in number when the nonholonomic constraint (2.1) is included.

The contribution of wheel input torques to the nonholonomic generalized active forces, $(\tilde{F}_r)_{w_j}$, will be calculated in the local wheel frame $\{w_j\}$, where expressions acquire their simpler form. Then the partial angular velocities of a wheel in its own frame will be

$$\tilde{\omega}_1^{w_j} = \frac{\cos(\phi^{w_j} + \theta)}{r^{w_j}}\mathbf{y}^{w_j} - \frac{\sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \times \left[\frac{{}^v l_x^{w_j} \sin \phi^{w_j} - {}^v l_y^{w_j} \cos \phi^{w_j}}{r^{w_j}}\mathbf{y}^{w_j} + \left(1 + \frac{\partial \phi^{w_j}}{\partial \theta}\right)\mathbf{z}^{w_j}\right] \tag{2.6a}$$

$$\tilde{\omega}_2^{w_j} = \frac{\sin(\phi^{w_j} + \theta)}{r^{w_j}}\mathbf{y}^{w_j} + \frac{\cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \times \left[\frac{{}^v l_x^{w_j} \sin \phi^{w_j} - {}^v l_y^{w_j} \sin \phi^{w_j}}{r^{w_j}}\mathbf{y}^{w_j} + \left(1 + \frac{\partial \phi^{w_j}}{\partial \theta}\right)\mathbf{z}^{w_j}\right] \tag{2.6b}$$

where the tilde is used to denote the *nonholonomic* partial angular velocities (see the Appendix). The nonholonomic partial angular velocities are less in number than normal holonomic partial velocities, because nonholonomic constraint equations have been utilized to eliminate some generalized speeds. The difference in number is equal to the number of nonholonomic constraints. In our case, one generalized speed, u_{w_j}, which was eliminated shortly after its introduction, was not included in the original set, so the number was reduced at the beginning by 1. Using Eq. (2.1), the number of generalized speeds was reduced to $n-1$. Nonholonomic partial angular velocities of wheel w_j for $r = 3, \dots, n-1$ are zero.

Steering torques have only a component in the $\mathbf{z}^{w_j}$ direction, whereas driving torques appear in the $\mathbf{y}^{w_j}$:

$$\mathbf{T}_s^{w_j} = \tau_s^{w_j}\mathbf{z}^{w_j} \tag{2.7a}$$

$$\mathbf{T}_d^{w_j} = \tau_d^{w_j}\mathbf{y}^{w_j} \tag{2.7b}$$

The contribution of these input torques to the nonholonomic generalized active forces is

$$\left(\tilde{F}_r\right)_{w_j} = \tilde{\boldsymbol{\omega}}_r^{w_j}\cdot\mathbf{T}_s^{w_j} + \tilde{\boldsymbol{\omega}}_r^{w_j}\cdot\mathbf{T}_d^{w_j} \qquad r = 1, \dots, n-1$$

and after some simple algebraic manipulation,

$$\begin{aligned}\left(\tilde{F}_1\right)_{w_j} = & -\frac{\sin(\phi^{w_j}+\theta)(1+(\partial\phi^{w_j}/\partial\theta))}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\tau_s^{w_j} \\ & + \frac{\cos(\phi^{w_j}+\theta)}{r^{w_j}}\tau_d^{w_j} \\ & + \frac{\sin(\phi^{w_j}+\theta)\left({}^v l_x^{w_j}\sin\phi^{w_j} - {}^v l_y^{w_j}\cos\phi^{w_j}\right)}{r^{w_j}\left({}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}\right)} \\ & \times \tau_d^{w_j} \quad \text{by (2.6) and (2.7)}\end{aligned} \tag{2.8}$$

$$\begin{aligned}\left(\tilde{F}_2\right)_{w_j} = & \frac{\cos(\phi^{w_j}+\theta)(1+(\partial\phi^{w_j}/\theta))}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\tau_s^{w_j} \\ & + \frac{\sin(\phi^{w_j}+\theta)}{r^{w_j}}\tau_d^{w_j} \\ & + \frac{\cos(\phi^{w_j}+\theta)\left({}^v l_x^{w_j}\sin\phi^{w_j} - {}^v l_y^{w_j}\cos\phi^{w_j}\right)}{r^{w_j}\left({}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}\right)} \\ & \times \tau_d^{w_j} \quad \text{by (2.6) and (2.7)}\end{aligned} \tag{2.9}$$

Generalized active forces for $r = 3, \dots, n-1$ are zero, since the corresponding partial angular velocities are zero. Since the wheel mass is neglected, no inertial forces or torques will be taken into account.

2.3. The Vehicle Body

The vehicle body has three degrees of freedom: two of them correspond to its position on the horizontal plane and the third to its orientation. These degrees of freedom are kinematically coupled with the wheels' rotation and steering angles. In fact, the wheels' kinematic quantities determine the vehicle body and degrees of freedom. The vehicle body motion is completely dependent on the motion of its wheels.

On the vehicle body, the following contact and field forces and torques can be identified:

1. The reaction forces by the wheels.
2. A force and torque by the attached manipulator.
3. The vehicle weight.
4. The reactions to the driving and steering torques.

Among these, the reaction forces by the wheels, the force and torque applied by the manipulator, and the weight of the vehicle are not contributing to the system generalized active forces.

The *forces applied by the wheels* are eliminated with their opposite due to the law of action and reaction. Since the interacting bodies do not develop linear velocity relative to each other, the products of partial velocities by the interaction forces are opposite.

Reaction forces and torques by the attached manipulator are noncontributing for the same reasons, but are necessary in the derivation of stability constraint conditions for the vehicle. Since they are noncontributing, they disappear from the equations and no information about them can be extracted. Therefore, we will apply a feature of Kane's approach that brings noncontributing forces into evidence by augmenting the vector of generalized speeds, without affecting the generalized coordinates. Specifically, we will theoretically allow the base of the attached manipulator to move freely. This will yield additional equations which will give expressions for the interaction forces.

The *vehicle weight* does not contribute either because the mass center of the vehicle translates with horizontal velocity. The resulting partial velocity is also horizontal and the dot product with the weight vanishes. This leaves the *reaction torques by the wheels* as the only contributing forces and torques.

We proceed with the calculation of the contributions of the reaction torques by the wheels and the interaction force and torque between the vehicle and the manipulator.

The vehicle mass center c_v, where frame $\{v\}$ is located, moves with velocity

$$\mathbf{v}^{c_v} = u_1 \mathbf{x}^I + u_2 \mathbf{y}^I \tag{2.10}$$

The corresponding nonholonomic partial velocities are

$$\tilde{\mathbf{v}}_1^{c_v} = \mathbf{x}^I \tag{2.11}$$

$$\tilde{\mathbf{v}}_2^{c_v} = \mathbf{y}^I \tag{2.12}$$

The mass center angular velocity is

$$\boldsymbol{\omega}^v = u_3 \mathbf{z}^I = \frac{u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \quad \text{by (2.1)} \tag{2.13}$$

Let $\overline{w_j}$ be the point of the vehicle body which coincides with the origin of frame $\{w_j\}$. Then this point will also have angular velocity

$$\boldsymbol{\omega}^{\overline{w_j}} = \frac{u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I$$

Its nonholonomic partial angular velocities will be

$$\tilde{\boldsymbol{\omega}}^{\overline{w_j}} = -\frac{\sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I = \tilde{\boldsymbol{\omega}}_1^v \tag{2.14}$$

$$\tilde{\boldsymbol{\omega}}_2^{\overline{w_j}} = \frac{\cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I = \tilde{\boldsymbol{\omega}}_2^v \tag{2.15}$$

and the contribution of the *reaction torques by the wheels* to the nonholonomic generalized active forces will be calculated as

$$\left(\tilde{F}_r\right)_{\overline{w_j}} = \tilde{\boldsymbol{\omega}}_r^{\overline{w_j}}(-\mathbf{T}_s^{w_j}) + \tilde{\boldsymbol{\omega}}_r^{\overline{w_j}}(-\mathbf{T}_d^{w_j})$$

which after substitution becomes

$$\left(\tilde{F}_1\right)_{\overline{w_j}} = \frac{\sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \tau_s^{w_j} \quad \text{by (2.7) and (2.14)} \tag{2.16}$$

$$\left(\tilde{F}_2\right)_{\overline{w_j}} = -\frac{\cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \tau_s^{w_j} \quad \text{by (2.7) and (2.15)} \tag{2.17}$$

The velocity of point $\overline{mm}$, at the point at which the manipulator is attached as on the vehicle, is calculated

$$\begin{aligned} \mathbf{v}^{\overline{mm}} &= \mathbf{v}^v + \boldsymbol{\omega}^v \times {}^v\mathbf{r}^{\overline{mm}} \\ &= u_1 \mathbf{x}^I + u_2 \mathbf{y}^I \\ &\quad + \frac{u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \times {}^v\mathbf{r}^{\overline{mm}} \end{aligned} \quad \text{by (2.10) and (2.13)} \tag{2.18}$$

and its nonholonomic partial velocities are

$$\tilde{\mathbf{v}}_1^{\overline{mm}} = \mathbf{x}^I - \frac{\sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \times {}^v\mathbf{r}^{\overline{mm}} \tag{2.19}$$

$$\tilde{\mathbf{v}}_2^{\overline{mm}} = \mathbf{y}^I + \frac{\cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \times {}^v\mathbf{r}^{\overline{mm}} \tag{2.20}$$

The points' angular velocity coincides with the vehicle angular velocity, expressed in Eq. (2.13). The partial angular velocities can be obtained from Eqs. (2.14) and (2.15).

If we denote by $\mathbf{F}^{\overline{mm}}$ and $\mathbf{T}^{\overline{mm}}$ the *force and torque exerted by the attached manipulator* on the vehicle, respectively, then their contribution to the generalized active forces is

$$\begin{aligned} \left(\tilde{F}_1\right)_{\overline{mm}} &= \left(\mathbf{x}^I - \frac{\sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \times {}^v\mathbf{r}^{\overline{mm}}\right) \\ &\quad \cdot \mathbf{F}^{\overline{mm}} \\ &\quad - \frac{\sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \cdot \mathbf{T}^{\overline{mm}} \end{aligned} \quad \text{by (2.14) and (2.19)} \tag{2.21}$$

$$\begin{aligned} \left(\tilde{F}_2\right)_{\overline{mm}} &= \left(\mathbf{y}^I + \frac{\cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \times {}^v\mathbf{r}^{\overline{mm}}\right) \\ &\quad \cdot \mathbf{F}^{\overline{mm}} \\ &\quad + \frac{\cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \cdot \mathbf{T}^{\overline{mm}} \end{aligned} \quad \text{by (2.15) and (2.20)} \tag{2.22}$$

Inertial forces and moments are exerted on the vehicle body. If the acceleration of the mass center of the vehicle, c_v, is

$$\mathbf{a}^{c_v} = \ddot{x}^v \mathbf{x}^I + \ddot{y}^v \mathbf{y}^I \tag{2.23}$$

the the inertial force is

$$\mathbf{R}^{*v} = -m^v(\ddot{x}^v \mathbf{x}^I + \ddot{y}^v \mathbf{y}^I) \tag{2.24}$$

The inertia moment exerted on the vehicle is

$$\mathbf{T}^{*c_v} = -\boldsymbol{\alpha}^v \cdot \mathbf{I}^v - \boldsymbol{\omega}^v \times \mathbf{I}^v \cdot \boldsymbol{\omega}^v \tag{2.25}$$

where $\boldsymbol{\alpha}^v$ is the angular acceleration of the vehicle,

$$\boldsymbol{\alpha}^v = \ddot{\theta}\mathbf{z}^I = \frac{d}{dt}\left\{\frac{u_2\cos(\phi^{w_j}+\theta) - u_1 \sin(\phi^{w_j}+\theta)}{{}^v l_y^{w_j} \sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\right\}\mathbf{z}^I \tag{2.26}$$

and $\mathbf{I}^v$ is the central inertia dyadic of the vehicle (see the Appendix). If, however, the principal moments of inertia I_1^v, I_2^v, and I_3^v in the central principal inertial frame $\{v_{\text{cpi}}\}$ are known and if one defines

$$\alpha_x^v \overset{\Delta}{=} \boldsymbol{\alpha}^v \cdot \mathbf{x}^{v_{\text{cpi}}} \qquad \omega_x^v \overset{\Delta}{=} \boldsymbol{\omega}^v \cdot \mathbf{x}^{v_{\text{cpi}}}$$
$$\alpha_y^v \overset{\Delta}{=} \boldsymbol{\alpha}^v \cdot \mathbf{y}^{v_{\text{cpi}}} \qquad \omega_y^v \overset{\Delta}{=} \boldsymbol{\omega}^v \cdot \mathbf{y}^{v_{\text{cpi}}}$$
$$\alpha_z^v \overset{\Delta}{=} \boldsymbol{\alpha}^v \cdot \mathbf{z}^{v_{\text{cpi}}} \qquad \omega_z^v \overset{\Delta}{=} \boldsymbol{\omega}^v \cdot \mathbf{z}^{v_{\text{cpi}}}$$

then the inertia moment can be expressed as[18]

$$\begin{aligned}\mathbf{T}^{*v} = &-\left[\alpha_x^v I_x^v - \omega_y^v\omega_z^v(I_y^v - I_z^v)\right]\mathbf{x}^{v_{\text{cpi}}} \\ &-\left[\alpha_y^v I_y^v - \omega_z^v\omega_x^v(I_z^v - I_x^v)\right]\mathbf{y}^{v_{\text{cpi}}} \\ &-\left[\alpha_z^v I_z^v - \omega_x^v\omega_y^v(I_x^v - I_y^v)\right]\mathbf{z}^{v_{\text{cpi}}}\end{aligned} \tag{2.27}$$

The contribution of the vehicle inertial forces and moments to the nonholonomic generalized inertial forces is calculated as

$$(\tilde{F}_r^*)_b = \tilde{\boldsymbol{\omega}}_r^v \cdot \mathbf{T}^{*v} + \tilde{\mathbf{v}}_r^{c_v} \cdot \mathbf{R}^{*v}$$

and after substitution,

$$\begin{aligned}(\tilde{F}_1^*)_b = &\frac{\sin(\phi^{w_j}+\theta)}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}} \\ &\times\Big\{\left[\alpha_x^v I_x^v - \omega_y^v\omega_z^v(I_y^v - I_z^v)\right]\mathbf{x}^{v_{\text{cpi}}}\cdot\mathbf{z}^I \\ &-\left[\alpha_y^v I_y^v - \omega_z^v\omega_x^v(I_z^v - I_x^v)\right]\mathbf{y}^{v_{\text{cpi}}}\cdot\mathbf{z}^I \\ &-\left[\alpha_z^v I_z^v - \omega_x^v\omega_y^v(I_x^v - I_y^v)\right]\mathbf{z}^{v_{\text{cpi}}}\cdot\mathbf{z}^I\Big\} - m^v\ddot{x}^v\end{aligned}$$

by (2.11), (2.14), (2.24), and (2.27) (2.28)

$$\begin{aligned}(\tilde{F}_2^*)_b = &-\frac{\cos(\phi^{w_j}+\theta)}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}} \\ &\times\Big\{\left[\alpha_x^v I_x^v - \omega_y^v\omega_z^v(I_y^v - I_z^v)\right]\mathbf{x}^{v_{\text{cpi}}}\cdot\mathbf{z}^I \\ &-\left[\alpha_y^v I_y^v - \omega_z^v\omega_x^v(I_z^v - I_x^v)\right]\mathbf{y}^{v_{\text{cpi}}}\cdot\mathbf{z}^I \\ &-\left[\alpha_z^v I_z^v - \omega_x^v\omega_y^v(I_x^v - I_y^v)\right]\mathbf{z}^{v_{\text{cpi}}}\cdot\mathbf{z}^I\Big\} - m^v\ddot{y}^v\end{aligned}$$

by (2.12), (2.15), (2.24), and (2.27) (2.29)

The vehicle total (including the wheels) contribution to the nonholonomic generalized active forces is found by simple addition of Eqs. (2.8), (2.16), and (2.21) and Eqs. (2.9), (2.17), and (2.22), respectively:

$$\begin{aligned}(\tilde{F}_1)_v = &-\frac{\sin(\phi^{w_j}+\theta)(\partial\phi^{w_j}/\partial\theta)}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\tau_s^{w_j} \\ &+\frac{\sin(\phi^{w_j}+\theta)\left({}^v l_x^{w_j}\sin\phi^{w_j} - {}^v l_y^{w_j}\cos\phi^{w_j}\right)}{r^{w_j}\left({}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}\right)}\tau_d^{w_j} \\ &+\frac{\cos(\phi^{w_j}+\theta)}{r^{w_j}}\tau_d^{w_j} \\ &+\left(\mathbf{x}^I - \frac{\sin(\phi^{w_j}+\theta)}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \times {}^v\mathbf{r}^{\overline{mm}}\right) \\ &\cdot\mathbf{F}^{\overline{mm}} \\ &-\frac{\sin(\phi^{w_j}+\theta)}{{}^v l^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I\cdot\mathbf{T}^{\overline{mm}}\end{aligned} \tag{2.30}$$

$$\begin{aligned}(\tilde{F}_2)_v = &\frac{\cos(\phi^{w_j}+\theta)(\partial\phi^{w_j}/\partial\theta)}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\tau_s^{w_j} \\ &+\frac{\cos(\phi^{w_j}+\theta)\left({}^v l_x^{w_j}\sin\phi^{w_j} - {}^v l_y^{w_j}\cos\phi^{w_j}\right)}{r^{w_j}\left({}^v l_y^{w_j}\sin\phi_x^{w_j}\cos\phi^{w_j}\right)}\tau_d^{w_j} \\ &+\frac{\sin(\phi^{w_j}+\theta)}{r^{w_j}}\tau_d^{w_j} \\ &+\left(\mathbf{y}^I + \frac{\cos(\phi^{w_j}+\theta)}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \times {}^v\mathbf{r}^{\overline{mm}}\right) \\ &\cdot\mathbf{F}^{\overline{mm}} \\ &+\frac{\cos(\phi^{w_j}+\theta)}{{}^v l_y^{w_j}\sin\phi^{w_j} + {}^v l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I\cdot\mathbf{T}^{\overline{mm}}\end{aligned} \tag{2.31}$$

2.4. The Manipulator

2.4.1. Link 0

Link 0 is the basis of the manipulator. It is rigidly attached on the mobile platform and moves along with it. Let *mm* be the point on the link at which it is connected to the vehicle.

As will be seen later on, the iteration forces and torques exerted on the vehicle by the manipulator are of great importance. These forces and torques do not contribute to the dynamic equations and do not appear in them. However, it is necessary to have an expression for them to be able to formulate stability conditions for the vehicle that carries the manipulator. For this reason, we make use of a feature of Kane's approach, namely, the introduction of additional generalized speeds to bring certain noncontributing forces into evidence.

In this framework, we let points have velocities and let rigid bodies have angular velocities which in fact they cannot. In this way, noncontributing forces and torques will appear in the augmented dynamics. Therefore, we let point *mm* translate and rotate in all directions w.r.t. the vehicle, so that its velocity will be

$$\begin{aligned}\mathbf{v}^{mm} &= \mathbf{v}^{\overline{mm}} + {}^{v}\mathbf{v}^{mm} \\ &= u_1\mathbf{x}^I + u_2\mathbf{y}^I \\ &\quad + \frac{u_2\cos(\phi^{w_j}+\theta) - u_1\sin(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \\ &\quad \times {}^{v}r^{\overline{mm}} + u_n\mathbf{x}^v + u_{n+1}\mathbf{y}^v + u_{n+2}\mathbf{z}^v \\ &\qquad \text{by (2.18)} \quad (2.32)\end{aligned}$$

and the angular velocity will be

$$\begin{aligned}\boldsymbol{\omega}^0 = \boldsymbol{\omega}^v + {}^{v}\boldsymbol{\omega}^0 &= \frac{u_2\cos(\phi^{w_j}+\theta) - u_1\sin(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \\ &\quad + u_{n+3}\mathbf{x}^v + u_{n+4}\mathbf{y}^v + u_{n+5}\mathbf{z}^v \\ &\qquad \text{by (2.13)} \quad (2.33)\end{aligned}$$

The nonholonomic partial velocities and partial angular velocities are therefore

$$\tilde{\mathbf{v}}_1^{mm} = \mathbf{x}^I - \frac{\sin(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \times {}^{v}\mathbf{r}^{\overline{mm}}$$

$$\tilde{\boldsymbol{\omega}}_1^0 = -\frac{\sin(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \quad (2.34)$$

$$\tilde{\mathbf{v}}_2^{mm} = \mathbf{y}^I + \frac{\cos(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \times {}^{v}\mathbf{r}^{\overline{mm}}$$

$$\tilde{\boldsymbol{\omega}}_2^0 = \frac{\cos(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \quad (2.35)$$

for $r = 1, 2$ and for the rest,

$$\tilde{\mathbf{v}}_n^{mm} = \mathbf{x}^v \qquad \tilde{\boldsymbol{\omega}}_{n+3}^0 = \mathbf{x}^v \quad (2.36a)$$

$$\tilde{\mathbf{v}}_{n+1}^{mm} = \mathbf{y}^v \qquad \tilde{\boldsymbol{\omega}}_{n+4}^0 = \mathbf{y}^v \quad (2.36b)$$

$$\tilde{\mathbf{v}}_{n+2}^{mm} = \mathbf{z}^v \qquad \tilde{\boldsymbol{\omega}}_{n+5}^0 = \mathbf{z}^v \quad (2.36c)$$

Thus, the contribution of the reaction forces from the platform to the nonholonomic generalized active forces is

$$(\tilde{F})_r^{mm} = \tilde{\mathbf{v}}_r^{mm} \cdot (-\mathbf{F}^{\overline{mm}}) + \tilde{\boldsymbol{\omega}}_r^0 \cdot (-\mathbf{T}^{\overline{mm}}) \qquad r = 1, 2, n-1, \ldots, n+5 \quad (2.37)$$

Given that link 0 can translate and rotate freely, its weight will contribute. The velocity of its mass center will be

$$\mathbf{v}^{c_0} = \mathbf{v}^{mm} + \boldsymbol{\omega}^0 \times ({}^{0}\mathbf{r}^{c_0} - {}^{0}\mathbf{r}^{mm})$$

where ${}^{0}\mathbf{r}^{c_0}$ is the position vector of mass center of link 0 w.r.t frame {0} and ${}^{0}\mathbf{r}^{mm}$ is the position vector of point *mm* w.r.t. the same frame. The above expression can be analyzed as

$$\begin{aligned}\mathbf{v}^{c_0} &= u_1\mathbf{x}^I + u_2\mathbf{y}^I \\ &\quad + \frac{U_2\cos(\phi^{w_j}+\theta) - u_1\sin(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \\ &\quad \times {}^{v}\mathbf{r}^{\overline{mm}} + u_n\mathbf{x}^v + u_{n+1}\mathbf{y}^v + u_{n+2}\mathbf{z}^v \\ &\quad + \Bigg(\frac{u_2\cos(\phi^{w_j}+\theta) - u_1\sin(\phi^{w_j}+\theta)}{{}^{v}l_y^{w_j}\sin\phi^{w_j} + {}^{v}l_x^{w_j}\cos\phi^{w_j}}\mathbf{z}^I \\ &\quad + u_{n+3}\mathbf{x}^v + u_{n+4}\mathbf{y}^v + u_{n+5}\mathbf{z}^v\Bigg) \\ &\quad \times ({}^{0}\mathbf{r}^{c_0} - {}^{0}\mathbf{r}^{mm}) \\ &\qquad \text{by (2.32) and (2.33)}\end{aligned}$$

Then the nonholonomic partial velocities can be derived:

$$\tilde{\mathbf{v}}_1^{c_0} = \mathbf{x}^I - \frac{\sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \times ({}^v \overline{\mathbf{r}^{mm}} + {}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm}) \quad (2.38)$$

$$\tilde{\mathbf{v}}_2^{c_0} = \mathbf{y}^I + \frac{\cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \mathbf{z}^I \times ({}^v \overline{\mathbf{r}^{mm}} + {}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm}) \quad (2.39)$$

$$\tilde{\mathbf{v}}_n^{c_0} = \mathbf{x}^v \qquad \tilde{\mathbf{v}}_{n+3}^{c_0} = \mathbf{x}^v \times ({}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm})$$
$$\tilde{\mathbf{v}}_{n+1}^{c_0} = \mathbf{y}^v \qquad \tilde{\mathbf{v}}_{n+4}^{c_0} = \mathbf{y}^v \times ({}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm}) \quad (2.40)$$
$$\tilde{\mathbf{v}}_{n+3}^{c_0} = \mathbf{z}^v \qquad \tilde{\mathbf{v}}_{n+5}^{c_0} = \mathbf{z}^v \times ({}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm})$$

Hence, the contribution of the link weight $\mathbf{G}^0 = m^0 g \mathbf{z}^I$ to the nonholonomic generalized active forces will be

$$\left(\tilde{F}_1\right)_{G0} = - \frac{m^0 g \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \times [\mathbf{z}^I \times ({}^v \overline{\mathbf{r}^{mm}} + {}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm})] \cdot \mathbf{z}^I \quad \text{by (2.38)} \quad (2.41)$$

$$\left(\tilde{F}_2\right)_{G0} = \frac{m^0 g \cos(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \times [\mathbf{z}^I \times ({}^v \overline{\mathbf{r}^{mm}} + {}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm})] \cdot \mathbf{z}^I \quad \text{by (2.39)} \quad (2.42)$$

$$\left(\tilde{F}_{n-3}\right)_{G0} = 0 \quad \text{by (2.40)}$$
$$\left(\tilde{F}_n\right)_{G0} = m^0 g [\mathbf{x}^v \times ({}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm})] \cdot \mathbf{z}^I \quad \text{by (2.40)}$$
$$\left(\tilde{F}_{n-2}\right)_{G0} = 0 \quad \text{by (2.40)}$$
$$\left(\tilde{F}_{n+1}\right)_{G0} = m^0 g \left[\mathbf{y}^v \times ({}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm})\right] \cdot \mathbf{z}^I \quad \text{by (2.40)}$$
$$\left(\tilde{F}_{n-1}\right)_{G0} = m^0 g \quad \text{by (2.40)}$$
$$\left(\tilde{F}_{n+2}\right)_{G0} = m^0 g [\mathbf{z}^v \times ({}^0\mathbf{r}^{c_0} - {}^0\mathbf{r}^{mm})] \cdot \mathbf{z}^I \quad \text{by (2.40)}$$

Instead of calculating the contribution of the reaction torque by the following link, we make the following general remark which can be applied to all the links of the serial manipulator. As can be seen from Eqs. (2.30) and (2.31), when the contribution of forces or torques and their reactions are added, the result is a contribution of the same force or torque multiplied by the partial relative velocity of the two interacting bodies. If there is no relative motion, the contribution is zero. Taking that as an example, we can calculate all the contributions of the manipulator joint torques in the same way,

$$\left(\tilde{F}_r\right)_{\text{joint } i} = \frac{\partial(\boldsymbol{\omega}^i - \boldsymbol{\omega}^{i-1})}{\partial u_r} \cdot \tau_i \mathbf{z}^{i-1} \quad (2.43)$$

where $\mathbf{z}^{i-1}$ is the axis of rotation link i w.r.t. link $i-1$, according to the Denavit-Hartenberg convention.

It should be noted that if one wishes to obtain the actual velocities of a point, all that has to be done is to set the additional generalized speeds in the expressions of velocities to zero. These additional generalized speeds are used only for the derivation of partial velocities and partial angular velocities. Inertial forces, the calculation of which is based on accelerations, are obtained by use of the actual accelerations. Thus, for the calculation of link 0 inertial forces and moments contribution, one proceeds as follows:

$$\begin{aligned}
\mathbf{a}^{c_0} &= \mathbf{a}^v + \boldsymbol{\omega}^v \times (\boldsymbol{\omega}^v \times {}^v\mathbf{r}^{c_0}) + \boldsymbol{\alpha}^v \times {}^v\mathbf{r}^{c_0} \\
&= \ddot{x}^v \mathbf{x}^I + \ddot{y}^v \mathbf{y}^I \\
&\quad + \left\{ \frac{u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \right\}^2 \mathbf{z}^I \\
&\quad \times [\mathbf{z}^I \times ({}^v\mathbf{r}^{mm} + {}^{mm}\mathbf{r}^{c_0})] + \ddot{\theta} \mathbf{z}^I \times ({}^v\mathbf{r}^{mm} + {}^{mm}\mathbf{r}^{c_0}) \\
&\quad \text{by (2.23), (2.13), and (2.26)}
\end{aligned}$$

$$\boldsymbol{\alpha}^{c_0} = \boldsymbol{\alpha}^v = \frac{d}{dt} \left\{ \frac{u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}} \right\} \mathbf{z}^I$$

Then the inertial forces can be found as

$$\mathbf{F}^{*0} = -m^0 \mathbf{a}^{c_0}$$

and the inertial moment $\mathbf{T}^{*0}$ is found in the same way as in Eq. (2.27):

$$\begin{aligned}
\mathbf{T}^{*0} = &-\left[\alpha_x^0 I_x^0 - \omega_y^0 \omega_z^0 \left(I_y^0 - I_z^0\right)\right] \mathbf{x}^{0(\text{cpi})} \\
&-\left[\alpha_y^0 I_y^0 - \omega_z^0 \omega_x^0 (I_z^0 - I_x^0)\right] \mathbf{y}^{0(\text{cpi})} \\
&-\left[\alpha_z^0 I_z^0 - \omega_x^0 \omega_y^0 \left(I_x^0 - I_y^0\right)\right] \mathbf{z}^{0(\text{cpi})}
\end{aligned}$$

This way the contribution of the inertia of link 0 can be found using the partial velocities and partial angular velocities in Eqs. (2.38), (2.39), (2.40), (2.34), (2.35), and (2.36):

$$(\tilde{F}^*)_0 = \tilde{\mathbf{v}}_r^{c_0} \cdot \mathbf{F}_0^* + \tilde{\boldsymbol{\omega}}_r^0 \cdot \mathbf{T}^{*0} \tag{2.44}$$

2.4.2. Links 1 to n

The rest of the mechanisms can be studied as usual. Each link i can rotate with simple angular velocity[18] w.r.t. the previous link $i-1$, that is, it rotates about axis $\mathbf{z}^{i-1}$. The points of interest for each link are the joint and its mass center. The angular velocity of the joint point $\boldsymbol{\omega}^i$,

$$\boldsymbol{\omega}^i = \boldsymbol{\omega}^{i-1} + {}^{i-1}\boldsymbol{\omega}^i = \boldsymbol{\omega}^{i-1} + \dot{q}^i \mathbf{z}^{i-1}$$

is needed for the calculation of the joint torque contribution, using Eq. (2.43). The velocity and the center of mass, $\mathbf{v}^{c_i}$,

$$\mathbf{v}^{c_i} = \mathbf{v}^i + \boldsymbol{\omega}^i \times {}^i\mathbf{r}^{c_i}$$
$$\mathbf{v}^i = \mathbf{v}^{i-1} + \boldsymbol{\omega}^i \times {}^{i-1}\mathbf{r}^i$$

are needed for the calculation of the contribution of the link weight. On the other hand, to obtain the inertial forces and moments exerted on the link, one must also calculate the acceleration, $\mathbf{a}^{c_i}$, and angular acceleration, $\boldsymbol{\alpha}^i$, of the mass center:

$$\mathbf{a}^i = \mathbf{a}^{i+1} + \boldsymbol{\alpha}^i \times {}^{i-1}\mathbf{r}^i + \boldsymbol{\omega}^i \times (\boldsymbol{\omega}^i \times {}^{i-1}\mathbf{r}^i)$$
$$\mathbf{a}^{c_i} = \mathbf{a}^i + \boldsymbol{\alpha}^i \times {}^i\mathbf{r}^{c_i} + \boldsymbol{\omega}^i \times (\boldsymbol{\omega}^i \times {}^i\mathbf{r}^{c_i})$$
$$\boldsymbol{\alpha}^i = \boldsymbol{\alpha}^{i-1} + \ddot{q}^i \mathbf{z}^{i-1} + \dot{q}^i \boldsymbol{\omega}^i \times \mathbf{z}^{i-1}$$

All these calculations can be performed with any recursive forward kinematics algorithm.[32] Since there is no specification on the number of links, analytical expressions will not be developed.

The links of the manipulator will make a contribution to the generalized active forces due to the action of the following:

1. The input torque at the joint.
2. The weight of each link.

The *input torque* contribution is calculated by Eq. (2.43) as

$$\left(\tilde{F}_r\right)_{\text{joint } i} = \frac{\partial(\boldsymbol{\omega}^i - \boldsymbol{\omega}^{i-1})}{\partial u_r} \cdot \tau_i \mathbf{z}^{i-1}$$

The contribution of the *weight* of link i will be calculated as

$$\left(\tilde{F}_r\right)_{Gi} = \frac{\partial \mathbf{v}^{c_i}}{\partial u_r} \cdot m^i \mathbf{g} \tag{2.45}$$

Inertial force on link i of the manipulator is expressed as

$$\mathbf{F}^{*i} = -m^i \mathbf{a}^{c_i}$$

and the inertial moment is expressed as

$$\mathbf{T}^{*i} = -\boldsymbol{\alpha}^i \cdot \mathbf{I}^i - \boldsymbol{\omega}^i \times \mathbf{I}^i \cdot \boldsymbol{\omega}^i$$

where $\mathbf{I}^i$ is the central inertial dyadic of link i. Then the contribution of inertial forces and moments of link i to generalized inertia forces is

$$\left(\tilde{F}_r^*\right)_i = \frac{\partial \mathbf{v}^{c_i}}{\partial u_r} \cdot \mathbf{F}^{*i} + \frac{\partial \boldsymbol{\omega}^i}{\partial u_r} \cdot \mathbf{T}^{*i}, \qquad r = 1, \ldots, n+5 \tag{2.46}$$

The interested reader, who wishes to see how Kane's equations can be developed for a serial manipulator, can refer to ref. 21, where the approach is applied to a Stanford manipulator.

2.4.3. The Effect of a Load

If a load is assumed at the end effector of the attached manipulator, then its effect can be replaced by a torque $\mathbf{T}^{\text{ee}}$ and a force $\mathbf{F}^{\text{ee}}$, applied at the origin of the end effector frame {ee}, which could be coincident with frame $\{n\}$. This torque and force make a contribution to the nonholonomic generalized active forces:

$$\left(\tilde{F}_r\right)_{\text{ee}} = \frac{\partial \mathbf{v}^{\text{ee}}}{\partial u_r} \cdot \mathbf{F}^{ee} + \frac{\partial \boldsymbol{\omega}^{\text{ee}}}{\partial u_r} \cdot \mathbf{T}^{\text{ee}} \tag{2.47}$$

2.5. Mobile Manipulator k Dynamic Equations

Kane's dynamic equations for mobile manipulator k can be formed as

$$\left(\tilde{F}_r\right)_k + \left(\tilde{F}_r^*\right)_k = 0 \qquad r = 1, \ldots, n+5 \tag{2.48}$$

where $(\tilde{F}_r)_k$ and $(\tilde{F}_r^*)_k$ are calculated as a sum of all nonholonomic generalized active forces and inertial forces, respectively, acting on the system. Specifically, $(\tilde{F}_r)_k$ is formed by summing the r terms $(r = 1, \ldots, n+5)$ appearing in Eqs. (2.30), (2.31), (2.37), (2.41), (2.42), (2.43), (2.45), and (2.47). On the other hand, $(\tilde{F}_r^*)_k$ is calculated as a sum of the r terms in Eqs. (2.28), (2.29), (2.44), and (2.46).

By the number of generalized coordinates, one should initially expect n equations. In fact, if one would like to analyze the system in more detail, he/she should include the generalized coordinates which correspond to each wheel rotation. Since this information is of no importance, we felt that it was preferable to spare these coordinates for the sake of reducing the dimension of the system. Therefore, we eliminated them immediately after their introduction through u_{w_j}, with the use of the available nonholonomic equations. By that time we were looking at n equations.

Using the no-skidding nonholonomic constraint, were were able to reduce the number of equations by 1. After the substitution of u_3 given by Eq. (2.1), $n-1$ equations remained. The final number of $n+5$ equations resulted from the need to bring the interacting forces between the vehicle and the attached manipulator into evidence. These forces could have been ignored, since they do not contribute to the actual equations. However, if one wishes to establish stability criteria for the mobile platform under the effect of load and accelerations, these forces are necessary.

Equation (2.48) is a reduced dynamics model for mobile manipulator k and should be accompanied by the nonholonomic constraint equation (2.1):

$$0 = \left(\tilde{F}_r\right)_k + \left(\tilde{F}_r^*\right)_k \qquad r = 1, \ldots, n+5$$

$$u_3 = \frac{u_2 \cos(\phi^{w_j} + \theta) - u_1 \sin(\phi^{w_j} + \theta)}{{}^v l_y^{w_j} \sin \phi^{w_j} + {}^v l_x^{w_j} \cos \phi^{w_j}}$$

Kane's equations can be rearranged to the form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) + \mathbf{F}_e = \boldsymbol{\tau} \qquad (2.49)$$

by noting that[3]

$$\mathbf{M}_{r,j} = \frac{\partial \sum_i F_r^{*i}}{\partial \ddot{q}_j}$$

$$\mathbf{C}_r(\mathbf{q}, \dot{\mathbf{q}}) = -\sum_i F_r^{*i} + \frac{\partial \sum_i F_r^{*i}}{\partial \ddot{q}_j} \ddot{q}_j$$

$$\mathbf{G}_r = -\sum_i \tilde{\mathbf{v}}^{c_i} \cdot \mathbf{G}^i$$

$$(\mathbf{F}_e)_r = -\left(\frac{\partial \mathbf{v}^{ee}}{\partial u_r} \cdot \mathbf{F}^{ee} + \frac{\partial \boldsymbol{\omega}^{ee}}{\partial u_r} \cdot \mathbf{T}^{ee} + \tilde{\mathbf{v}}_r^{mm}(-\overline{\mathbf{F}^{mm}}) + \boldsymbol{\omega}_r^0(-\overline{\mathbf{T}^{mm}}) \right)$$

$$\boldsymbol{\tau}_k = \tilde{\boldsymbol{\omega}}_r^{w_j} \cdot (\mathbf{T}_s^{w_j} + \mathbf{T}_d^{w_j}) + \frac{\partial(\boldsymbol{\omega}^i - \boldsymbol{\omega}^{i-1})}{\partial u_r} \cdot \tau_i \mathbf{z}^{i-1}$$

where $\mathbf{q}$ is the vector of generalized coordinates for mobile manipulator k, and r and j range from 1 to $n+5$. It should also be made clear that $\mathbf{C}$ does not depend on $\ddot{q}$: the term which contains the accelerations in the above defining equation eliminates all acceleration terms from $\sum_i F_r^{*i}$ so that only velocity dependent terms are left.

Let the operational point $\mathbf{p}$ of the mobile manipulator be a fixed point on the manipulator end effector ee. The mobile manipulator is a redundant system, so the coordinates of the operational point are insufficient to describe the full dynamics of the mobile manipulator. The dynamics of the operational point can be described though. Before we proceed, it is necessary to specify the parameters used in the representation of the orientation of frame {ee}, since different parameters can be chosen and the expressions may differ. In the special case where "*X-Y-Z* fixed angles"[33] are used, the Jacobian matrix associating the joint velocities to the operational point velocities is known as *geometrical*[32] or *basic*[19] Jacobian:

$$\begin{pmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{pmatrix} = \mathbf{J}_0(\mathbf{q})\dot{\mathbf{q}}$$

It has been shown by Khatib[19] that any Jacobian resulting from the use of different orientation parameters is related to the geometrical Jacobian:

$$\mathbf{J}(\mathbf{q}) = \mathbf{E}_p \mathbf{J}_0(\mathbf{q})$$

Another Jacobian can be derived by direct differentiation of the forward kinematics function and is called *analytical*. No matter which orientation representation is used, the differential kinematics relation can be generally expressed as

$$\dot{\mathbf{p}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

which after differentiation becomes

$$\ddot{\mathbf{p}} = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) \qquad (2.50)$$

Then, Eqs. (2.49) and (2.50) can be used to obtain the operational point dynamics

$$\boldsymbol{\Lambda}(\mathbf{q})\ddot{\mathbf{p}} + \boldsymbol{\mu}(\mathbf{q}, \dot{\mathbf{q}}) - \boldsymbol{\nu}(\mathbf{q}) = \mathbf{F}$$

where

$$\boldsymbol{\Lambda}(\mathbf{q}) = [\mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T]^{-1} \qquad \boldsymbol{\mu}(\mathbf{q},\dot{\mathbf{q}}) = \bar{\mathbf{J}}^T\mathbf{C} - \boldsymbol{\Lambda}\,\mathbf{h}$$

$$\boldsymbol{\nu}(\mathbf{q}) = \bar{\mathbf{J}}^T\mathbf{g} \qquad \mathbf{F} = \bar{\mathbf{J}}^T\boldsymbol{\tau}$$

and $\bar{\mathbf{J}}$ is the generalized inverse of $J(q_k)$, corresponding to the solution that minimizes joint velocities norms and is equal to[19]

$$\bar{\mathbf{J}}(\mathbf{q}) = \mathbf{M}^{-1}(\mathbf{q})\mathbf{J}^T(\mathbf{q})\boldsymbol{\Lambda}(\mathbf{q})$$

The analysis so far has focused on a single mobile manipulator and no distinction was made from any other mobile manipulator for reasons of clarity. Having completed our analysis of this subsystem, it is convenient to distinguish the dynamic equations of each mobile manipulator using a subscript k that refers to a specific mobile manipulator k:

$$\boldsymbol{\Lambda}_k(\mathbf{q}_k)\ddot{\mathbf{p}}_k + \boldsymbol{\mu}_k(\mathbf{q}_k,\dot{\mathbf{q}}_k) + \boldsymbol{\nu}_k(\mathbf{q}_k) = \mathbf{F}_k \tag{2.51}$$

3. MODELING THE DEFORMABLE OBJECT

A deformable object can have infinite degrees of freedom, each associated with a mass particle in the body. In a microscopic scale, the motion of a particle in a deformable object under external load is governed by the elastodynamic equations. Consider a mass particle in a deformable body of arbitrary shape, which is under the influence of external forces. The position of the particle w.r.t. the inertial coordinate frame is determined by its coordinates $(x \quad y \quad z)$. The displacement of the particle is represented by the vector

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \Delta x(x,y,z,t) \\ \Delta y(x,y,z,t) \\ \Delta z(x,y,z,t) \end{bmatrix}$$

where $\Delta x(x,y,z,t)$ is the displacement in the direction of the x axis, and $\Delta y(x,y,z,t)$ and $\Delta z(x, y,z,t)$ are the displacements in the directions of the y and z axes, respectively.

The body deformations are represented by the vector $\boldsymbol{\varepsilon}$, which is related to displacements as follows:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}}_{\mathbf{D}} \cdot \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\boldsymbol{\varepsilon}} = \mathbf{D}\cdot\boldsymbol{\varepsilon} \tag{3.1}$$

The deformations produce internal stresses in the body. Let E be the elasticity modulus and ν the Poisson ratio of the deformable material. The Lamé constants are

$$\mu = \frac{E}{2(1+\nu)} \qquad \lambda = 2\mu\frac{\nu}{1-2\nu}$$

The relation between stresses and deformations is expressed by the well known Hook law, which can be expressed as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \underbrace{\begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & & & \\ \lambda & 2\mu+\lambda & \lambda & & \mathbf{0}_3 & \\ \lambda & \lambda & 2\mu+\lambda & & & \\ & & & \mu & 0 & 0 \\ & \mathbf{0}_3 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix}}_{\mathbf{E}} \cdot\boldsymbol{\varepsilon} \tag{3.2}$$

where $\mathbf{0}_3$ is the 3×3 null matrix. This is the case where there are no remaining stresses or deformations in the body.

The equations of equilibrium for the material particle can be expressed as

$$\mathbf{D}^T\boldsymbol{\sigma} + \mathbf{f} = 0 \tag{3.3}$$

Substituting (3.1) and (3.2) into (3.3) yields the elastodynamic equations for the deformable body for the case of no remaining stresses and deformations,

$$\mathbf{D}^T\mathbf{E}\mathbf{D}\boldsymbol{\varepsilon} + \mathbf{f} = 0 \tag{3.4}$$

where $\mathbf{f}$ are the "external" forces acting on the body,

$$\mathbf{f} = \mathbf{f}_I + \mathbf{f}_d + \mathbf{f}_t + \mathbf{f}_g \tag{3.5}$$

where $\mathbf{f}_I$, $\mathbf{f}_d$, and $\mathbf{f}_t$ correspond to inertial, damping, and external forces, respectively. If viscous damping within the material is assumed, they will be

$$\mathbf{f}_I = -\rho\frac{\partial^2\boldsymbol{\varepsilon}}{\partial t^2},$$

$$\mathbf{f}_d = -c_d\frac{\partial\varepsilon}{\partial t} = -c_d\frac{\partial(\mathbf{D}\cdot\boldsymbol{\varepsilon})}{\partial t} \quad \text{by (3.1)}$$

$$\mathbf{f}_t = -\rho\mathbf{g} + \mathbf{f}_c$$

where $\mathbf{g}$ is the gravity acceleration, ρ is the object density, which is assumed constant, and $\mathbf{f}_c$ are the contact forces which other bodies exert on the object.

Equation (3.4) has to be integrated on the object volume, which can be done using finite elements. Integration of the equations yields the displacements of every point of the body and thus determines the final shape of the object and its strain distribution.

4. MULTIPLE MOBILE MANIPULATOR SYSTEM

4.1. Introduction

So far we have derived a model for one mobile manipulator and a deformable object. Suppose that there are m mobile manipulators grasping the object. In this section, the dynamic equations of the mobile manipulators will be combined with the use of the deformable body dynamic equations.

Due to the deformable nature of the common manipulated object, the operational point coordinates of one manipulator are not determined by the operational coordinates of other manipulators. The deformable object reacts to the relative motion of the grasp points, but does not restrict it. Thus the combined system degrees of freedom are not confined. On the contrary, if one also considers the deformable body's degrees of freedom, the combined degrees are increased to a theoretically infinite number. This is obviously not desirable from the control point of view.

Although the deformable object cannot be formally controlled, a limited number of its degrees of freedom, namely, the coordinates of the mobile manipulators' operational points, can be driven. A reasonable approach is to consider a global operational point that includes all controllable degrees of freedom. Bearing in mind that the motion of the deformable object can be calculated to a certain degree of accuracy given the deformations applied or the forces exerted on its outer surface (boundary conditions), we can regard the controllable degrees of freedom as independent and the rest as "semidependent." This motivates us to describe the dynamics of the deformable object in terms of the coordinates of the operational points. This approach renders the deformable object as a deformable lattice, the nodes of which are the operational points of the mobile manipulators (Fig. 3). The geometric parameters of the lattice can be chosen in such a way that the deformable rods in it have a total mass equal to that of the original object and that the mass distribution is such that the position of the mass centers of each construction does not differ significantly. A similar approach to rigid bodies under manipulation was followed in ref. 34.

Our approach calls for the definition of a global operational point, to which every mobile manipulator will make a contribution. These contributions are not completely decoupled, since each mobile manipulator interacts with the rest, exerting forces through the deformable object. The definition of a global operational point will follow. Then, having described all subsystems in terms of a single, global operational point, it is possible to utilize the *augmented object approach*.[22]

The whole system is considered as a closed-chain robotic system and is examined within the framework of such systems.

4.2. The Global Operational Space Concept

The state of the deformable object can only be determined when either the external forces or the displacement vector and its derivatives have been specified for all points of grasp. These serve as

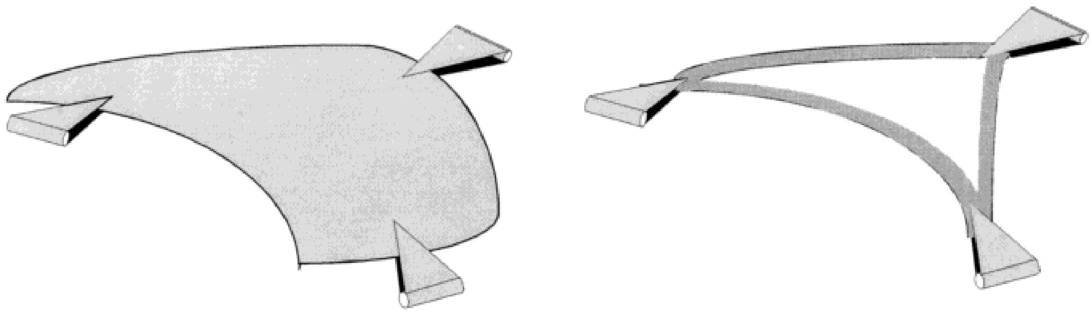

Figure 3. Approximating the deformable object.

boundary conditions for Eqs. (3.4). Let us define the global operational point:

Definition 1: The global operational point is a vector formed by the cartesian product of the operational points of all manipulator systems grasping the deformable object,

$$\mathbf{P} \stackrel{\Delta}{=} [\mathbf{p}^1 \quad \mathbf{p}^2 \quad \cdots \quad \mathbf{p}^k \quad \cdots \quad \mathbf{p}^m]^T$$
$$\in \mathscr{D}_{p^1} \times \mathscr{D}_{p^2} \times \cdots \times \mathscr{D}_{p^k} \times \cdots \mathscr{D}_{p^m}$$

where p^i is the operational point defined for each mobile manipulator k, the dynamics of which the model (2.51) describes. $\mathscr{D}_{p^i}$ is the operational space of mobile manipulator i and m is the total number of mobile manipulators in the system.

From Definition 1, it follows that the global operational space is the direct sum of the task space of the mobile manipulators grasping the object. Obviously, its dimension is also the sum of the degree of manipulability of each mobile manipulator. This is so because the motion of a particular grasp point is not restricted by the motion of the others; there is force interaction between them.

4.3. Generalization of the Augmented Object Approach

In the framework of the *augmented object approach*, only rigid objects have been considered. The definition of a global operational point permits the generalization of this methodology to the case of deformable objects as well. All subsystems can be combined into a single, centralized dynamic model, defined in the global operational space.

Vector **P**, along with its derivatives, provides the necessary boundary conditions for (3.4), by which the displacements of all points of the deformable body can be calculated, as well as the forces at the end effectors with which the object reacts to the manipulators.

A deformable object can have infinite degrees of freedom. Consequently, it is impossible to formally control all its degrees of freedom. In any case, the general behavior of a deformable object will defy accurate mathematical modeling and the question lies on the degree of desired accuracy and on the conditions of operation that one is interested in. We believe that placing too much emphasis on accurately describing the behavior of the deformable object will unnecessarily complicate the model and will make control difficult. We propose the use of an approximating deformable lattice (Fig. 3), the nodes of which are the grasp points of the mobile manipulators involved. On such a deformable grid, one can apply simplified finite elements to obtain information about the state of the grasp points. Given this as a starting point, one can proceed to higher degrees of accuracy by refining the grid.

The Galerkin equations for (3.4) can be constructed as

$$\int_V \delta \boldsymbol{\varepsilon}^T [\mathbf{D}^T \mathbf{E} \mathbf{D} \boldsymbol{\varepsilon} + \mathbf{f}] \, dV = 0$$

With algebraic manipulation on the first term in the above integral, the equation becomes

$$\oint_S \delta \boldsymbol{\varepsilon}^T \, s \, dS - \int_V \delta \varepsilon^T \boldsymbol{\sigma} \, dV + \int_V \delta \boldsymbol{\varepsilon}^T \, \mathbf{f} \, dV = 0$$

where **s** is the resultant of all stresses on the boundary surface S of the object and is equal to $\mathbf{f}_c$.

Substituting (3.5) and rearranging the terms yields

$$\int_V \delta\boldsymbol{\varepsilon}^T \rho \frac{\partial^2 \boldsymbol{\varepsilon}}{\partial t^2}\, dV + \int_V \delta\boldsymbol{\varepsilon}^T c_d \frac{\partial(\mathbf{D}\cdot\boldsymbol{\varepsilon})}{\partial t}\, dV + \int_V \delta\varepsilon^T \boldsymbol{\sigma}\, dV + \delta\boldsymbol{\varepsilon}^T \rho\mathbf{g}\, dV = \oint_S \delta\boldsymbol{\varepsilon}^T \mathbf{f}_c\, dS + \int_V \delta\boldsymbol{\varepsilon}^T \rho\mathbf{g}\, dV \tag{4.1}$$

Observing the above equation, we can see the equivalence to the general expression of the equations of motion for mechanical systems:

$$\mathscr{M}(x)\ddot{x} + \mathscr{C}(x, \dot{x}) + \mathscr{K}(x) + \mathscr{P} = \mathscr{F}_e$$

Indeed, if the vector of degrees of freedom is defined as $\mathbf{a} = \mathbf{P}$, and the displacements of an object particle are approximated by a series $\boldsymbol{\varepsilon} = \mathbf{N}\mathbf{a}$, where $\mathbf{N}$ is a vector of known and linearly independent functions, as commonly done in finite element analysis, then the terms in Eq. (4.1) can take the form

$$\int_V \delta\boldsymbol{\varepsilon}^T \rho \frac{\partial^2 \boldsymbol{\varepsilon}}{\partial t^2}\, dV = \delta\mathbf{a}^T \underbrace{\int_V \rho\mathbf{N}^T\mathbf{N}\, dV}_{\mathscr{M}}\, \ddot{\mathbf{a}}$$

$$\int_V \delta\boldsymbol{\varepsilon}^T c_d \frac{\partial(\mathbf{D}\cdot\boldsymbol{\varepsilon})}{\partial t}\, dV = \delta\mathbf{a}^T \underbrace{\int_V \mathbf{N}^T\mathbf{D}\mathbf{N}\, dV}_{\mathscr{C}}\, \dot{\mathbf{a}}$$

$$\int_V \delta\varepsilon^T \boldsymbol{\sigma}\, dV = \delta\mathbf{a}^T \underbrace{\int_V \mathbf{N}^T\mathbf{D}^T\mathbf{E}\mathbf{D}\mathbf{N}\, dV}_{\mathscr{K}}\, \mathbf{a}$$

$$\int_V \delta\boldsymbol{\varepsilon}^T \rho\mathbf{g}\, dV = \delta\mathbf{a}^T \underbrace{\int_V \mathbf{N}^T\rho\mathbf{g}\, dV}_{\mathscr{P}}$$

$$\oint_S \delta\boldsymbol{\varepsilon}^T \mathbf{f}_c\, dS = \delta\mathbf{a}^T \underbrace{\oint_S \mathbf{N}^T\mathbf{f}_c\, dS}_{\mathscr{F}_e}$$

This way the dynamic equations of the deformable object are expressed as a function of the global operational point coordinates. Considering the m mobile manipulators in a single system with dynamic equations,

$$\mathrm{diag}(\boldsymbol{\Lambda}_i)\ddot{\mathbf{P}} + \mathrm{diag}(\boldsymbol{\mu}_i) + \mathrm{diag}(\boldsymbol{\nu}_i) = \mathrm{diag}(\mathbf{F}_i)$$

$$\Leftrightarrow \quad \boldsymbol{\Lambda}\ddot{\mathbf{P}} + \boldsymbol{\mu} + \boldsymbol{\nu} = \mathbf{F} \qquad i = 1, \ldots, m$$

Now the *augmented object*[22] approach can be generalized to yield the system's dynamic equations

$$\boldsymbol{\Lambda}_{\oplus}\ddot{\mathbf{P}} + \boldsymbol{\mu}_{\oplus} + \boldsymbol{\nu}_{\oplus} = \mathbf{F}_{\oplus} \tag{4.2}$$

where

$$\boldsymbol{\Lambda}_{\oplus} = \mathscr{M} + \boldsymbol{\Lambda} \qquad \boldsymbol{\mu}_{\oplus} = \mathscr{C} + \boldsymbol{\mu}$$

$$\boldsymbol{\nu}_{\oplus} = \mathscr{K} + \boldsymbol{\nu} \qquad \mathbf{F}_{\oplus} = \mathscr{F}_e + \mathbf{F}$$

and $\mathbf{P}$ is the global operational point which includes the modeled degrees of freedom of the deformable object.

5. DYNAMIC CONSTRAINTS

5.1. The Closed Chain Problem

All cooperating mobile manipulators form a closed chain through the ground. It is known that a wheeled type mobile robot can be transformed into a virtual manipulator[35] with unactuated joints,[23] based on the fact that manipulators with unactuated joints are often nonholonomic. Each of these virtual manipulators connects the platform with the ground, constraining each platform to a nonholonomic motion.

A common approach to treating closed chains is to cut the chain at an actuated or unactuated joint transforming the closed chain to an open one, and replace the joint with holonomic constraint equations.[25,26,30] The forces at the cut joints are calculated either with[26] or without Lagrange multipliers.[25] Following the same approach, we could cut the closed chain at the mobile platforms location and replace these ''virtual'' joints with the dynamic equations of the mobile platform. However, within Kane's framework, this is not necessary for two reasons:

1. The forces exerted from the ground to each mobile platform do not contribute to the dynamic equations and are not necessary to determine the motion of the system.
2. Constraint equations have already been included in the model [Eqs. (2.1) and (2.5)].

The latter establishes the equivalence of our treatment to existing approaches. The former indicates that this is an efficient way to handle the closed chain, since these reaction forces are generally difficult to resolve (practically, they can only be calculated when the mobile platform is modeled as a unicycle).

5.2. Constraints

Nonholonomic constraints have so far integrated within the model equations without the need for Lagrange multipliers. These can ensure that the vehicle theoretical trajectories are nonholonomic and that each platform velocity can be derived from the angular velocity of its wheels through the no-slipping condition. This is an approach that is followed by many researchers.[5,9] However, are these enough to guarantee that the vehicles do not slip or skid?

In practice, velocity conditions are only necessary and not sufficient, because they describe the result and not the cause: The constraint equations (2.1) and (2.5), describe the motion of a system under the effect of constraint forces, but do not ensure that these forces can really be applied. In this section, we treat this problem by the following actions:

- Completing the set of nonholonomic constraints (2.1) and (2.5) by additional expressions involving constraint reaction forces at the wheels: Eqs. (5.3) and (5.5).
- Specifying the domain of admissible values for the constraint forces: Eqs. (5.1), (5.2), and (5.4).
- Indicating how one can ensure that friction forces will not saturate and how tipping over can be avoided: Eqs. (5.3), (5.5), and (5.6).

It must be noted that these reaction forces from the ground cannot be calculated explicitly, since it turns out that the number of unknowns is larger than the given equations.[8] The ambiguity increases with the number of wheels.

In the analysis that follows, we will distinguish the quantities that refer to a specific mobile manipulator by a subscript k. First of all, the sum of all vertical ground reactions acting on the mobile platform of mobile manipulator k should counterbalance the vertical component of the resultant of all other contact and distant forces exerted on the platform. If we express this resultant as

$$\overline{\mathbf{R}}_k = m_k^v(\mathbf{g} - \mathbf{a}_k^v) + \mathbf{F}_k^{\overline{mm}}$$

and denote the reaction forces exerted on the wheels of the platform k by $\mathbf{F}_k^{w_j}$, then we can write for each mobile manipulator

$$\overline{\mathbf{R}}_k \mathbf{z}^I = \sum_{j=1}^{w} \mathbf{F}_k^{w_j} \mathbf{z}^I \qquad k = 1, \ldots, m \tag{5.1}$$

Moreover, the vertical components of reaction forces should have positive direction

$$0 \leq \mathbf{F}_k^{w_j} \mathbf{z}_k^{w_j} \qquad k = 1, \ldots, m \tag{5.2}$$

It is also true that the horizontal components of $\mathbf{F}_k^{w_j}$ should satisfy

$$r_k^{w_j} \mathbf{F}_k^{w_j} \times {}_k^{w_j} = \boldsymbol{\tau}_{d_k}^{w_j} \qquad k = 1, \ldots, m \tag{5.3}$$

For slipping not to occur, the condition of static friction must be satisfied,[36]

$$0 \geq \frac{1}{\sqrt{1+\eta}} \|\mathbf{F}_k^{w_j}\| - \mathbf{z}_k^{w_j} \mathbf{F}_k^{w_j} \qquad k = 1, \ldots, m \tag{5.4}$$

where η is the coefficient of static friction between the wheel and the ground, assumed equal for all vehicles. Since the vehicles perform an instantaneous circular motion, it follows that

$$\left(\overline{\mathbf{R}}_k + \sum_{j=1}^{w} \mathbf{F}_k^{w_j} \right) \cdot \boldsymbol{\omega}_k^v \times \mathbf{v}_k^{c_v} = \|\boldsymbol{\omega}_k^v \times \mathbf{v}_k^{c_v}\|^2 m_k^v$$
$$k = 1, \ldots, m \tag{5.5}$$

which means that the components of all external forces in a direction perpendicular to the vehicle trajectory should play the role of a centripetal force. The reader perhaps wonders why the nonholonomic constraint equation (2.1) is not sufficient on its own. The truth is that unless friction conditions are examined, the system may assume that everything is going as scheduled; however, there may not be adequate centripetal force to maintain the circular motion. This would result in performance deterioration and even stability problems in the case that no real-time localization and feedback compensation scheme is implemented. On the other hand, Eq. (5.5) is not sufficient for nonholonomy, since it is always satisfied, even when skidding occurs. Both of them however, make a set of sufficient conditions for the constraints being imposed. The same discussion applies to the pair of nonholonomic constraint equations (2.5) and (5.3).

Finally, the moments of all external forces exerted on mobile platform k are related by

$$0 = \mathbf{T}_k^{*v} + \mathbf{F}_k^{\overline{mm}} \times {}^v\mathbf{r}^{\overline{mm}}{}_k + \sum_{j=1}^{w} \mathbf{F}_k^{w_j} \times {}^v \boldsymbol{l}_k^{w_j}$$

$$\Rightarrow \quad 0 = -\boldsymbol{\alpha}_k^v \cdot \mathbf{I}_k^v - \boldsymbol{\omega}_k^v \times \mathbf{I}_k^v \cdot \boldsymbol{\omega}_k^v + \mathbf{F}_k^{\overline{mm}} \times {}^v\mathbf{r}_k^{\overline{mm}}$$
$$+ \sum_{j=1}^{w} \mathbf{F}_k^{w_j} \times {}^v \boldsymbol{l}_k^{w_j} \qquad k = 1, \ldots, m \tag{5.6}$$

where $\mathbf{I}_k^v$ is the central inertia dyadic of the platform of mobile manipulator k. The above condition can ensure that no tipping over will occur, given that $\boldsymbol{\omega}_k^v$ and $\boldsymbol{\alpha}_k^v$ have strictly vertical direction. That is, if the angular velocity $\boldsymbol{\omega}_k^v$ and angular acceleration of the vehicle $\boldsymbol{\alpha}_k^v$ take values on the $\mathbf{z}_k^v$ axis, then the reaction forces must satisfy the above matrix constraint equation.

Finally, one should not forget the constraints used already in the derivation of the dynamic equations. These are the no-skidding condition expressed in Eq. (2.1),

$$(u_3)_k = \frac{(u_2)_k \cos(\phi_k^{w_j} + \theta_k) - (u_1)_k \sin(\phi_k^{w_j} + \theta_k)}{\left({}^v l_y^{w_j}\right)_k \sin \phi^{w_j}{}_k + ({}^v l_x^{w_j})_k \cos \phi_k^{w_j}}$$

and the no-slipping condition in Eq. (2.5),

$$\begin{aligned}(u_{w_j})_k = \frac{1}{r_k^{w_j}}\Big[&(u_1)_k \cos(\phi_k^{w_j} + \theta_k) \\ &+ (u_2)_k \sin(\phi_k^{w_j} + \theta_k) \\ &+ (u_3)_k\Big(({}^v l_x^{w_j})_k \sin \phi_k^{w_j} - \left({}^v l_y^{w_j}\right)_k \cos \phi_k^{w_j}\Big)\Big]\end{aligned}$$

After its substitution to the generalized speeds expressions, Eq. (2.5) can serve to calculate the wheel angular velocity.

5.3. The Set of Possible Reaction Forces

With the exception of trivial cases, the reaction forces that the ground exerts on the wheels of each mobile platform cannot be calculated analytically, since there is a kind of redundancy for these forces. The reaction forces cannot be resolved and they are distributed to the wheels of each platform in an unknown way.

The constraints stated in the previous section define a set of solutions for the reaction forces. Gathering all constraints together in a set of algebraic expressions,

$$\overline{\mathbf{R}}_k \mathbf{z}^I = \sum_{j=1}^{w} \mathbf{F}_k^{w_j} \mathbf{z}^I$$

$$0 \le \mathbf{F}_k^{w_j} \mathbf{z}_k^{w_j}$$

$$0 \ge \frac{1}{\sqrt{1+\eta}} \|\mathbf{F}_k^{w_j}\| - \mathbf{z}_k^{w_j} \mathbf{F}_k^{w_j}$$

$$\tau_{d_k}^{w_j} = r_k^{w_j} \mathbf{F}_k^{w_j} \mathbf{x}_k^{w_j}$$

$$\left(\overline{\mathbf{R}}_k + \sum_{j=1}^{w} \mathbf{F}_k^{w_j}\right) \cdot \boldsymbol{\omega}_k^v \times \mathbf{v}_k^{c_v} = \|\boldsymbol{\omega}_k^v \times \mathbf{v}_k^{c_v}\|^2 m_k^v$$

$$0 = -\boldsymbol{\alpha}_k^v \cdot \mathbf{I}_k^v - \boldsymbol{\omega}_k^v \times \mathbf{I}_k^v + \overline{\mathbf{F}_k^{mm}} \times {}^v\overline{\mathbf{r}_k^{mm}} + \sum_{j=1}^{w} \mathbf{F}_k^{w_j} \times {}^v \boldsymbol{l}_k^{w_j}$$

$$(u_3)_k = \frac{(u_2)_k \cos(\phi_k^{w_j} + \theta_k) - (u_1)_k \sin(\phi_k^{w_j} + \theta_k)}{\left({}^v l_y^{w_j}\right)_k \sin \phi^{w_j}{}_k + ({}^v l_x^{w_j})_k \cos \phi_k^{w_j}}$$

$$\begin{aligned}(u_{w_j})_k = \frac{1}{r_k^{w_j}}\Big[&(u_1)_k \cos(\phi_k^{w_j} + \theta_k) \\ &+ (u_2)_k \sin(\phi_k^{w_j} + \theta_k) \\ &+ (u_3)_k\Big(({}^v l_x^{w_j})_k \sin \phi_k^{w_j} - \left({}^v l_y^{w_j}\right)_k \cos \phi_k^{w_j}\Big)\Big]\end{aligned}$$

we can investigate the structure of the set of admissible solutions for the reaction forces, which is defined by the above equations.

Control systems that are constrained are usually treated within the framework of optimal control. In view of the techniques used within this framework, it is important for the set of possible solutions defined by the constraint equations to be convex. This is necessary for the use of Kuhn–Tucker conditions on which many methodologies are based.

All of the above constraints that are linear in $\mathbf{F}_k^{w_j}$ are convex with respect to this vector variable. The only nonlinear constraint function is

$$0 \ge \frac{1}{\sqrt{1+\eta}} \|\mathbf{F}_k^{w_j}\| - \mathbf{z}_k^{w_j} \mathbf{F}_k^{w_j}$$

If we let

$$\left[(F_x^{w_j})_k \quad (F_y^{w_j})_k \quad (F^{w_j})_k\right]^T$$

be the coordinates of $\mathbf{F}_k^{w_j}$ in frame $\{w_j\}_k$, then the constraint function can take the form

$$0 \ge g \stackrel{\Delta}{=} \sqrt{(F_k^{w_j})_k^2 + (F_x^{w_j})_k^2} - \sqrt{1+\eta}\,(F_z^{w_j})_k \quad (5.7)$$

The region defined by (5.7) is the interior of the cone shown in Figure 4. It is obviously a convex region as

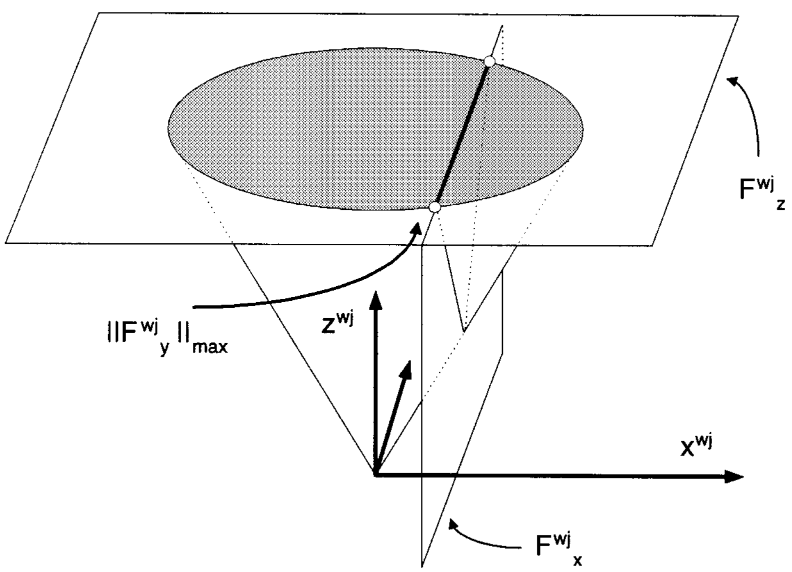

Figure 4. The friction cone and the admissible reaction forces.

can be seen by the Hessian matrix of g (subscript k and superscript w_j are dropped for reasons of simplicity in presentation),

$$H_g = \begin{bmatrix} \dfrac{\partial^2 g}{\partial F_x^2} & \dfrac{\partial^2 g}{\partial F_x\, \partial F_y} & \dfrac{\partial^2 g}{\partial F_x^2\, \partial F_z^2} \\ \dfrac{\partial^2 g}{\partial F_x\, \partial F_y} & \dfrac{\partial^2 g}{\partial F_y^2} & \dfrac{\partial^2 g}{\partial F_y\, \partial F_z} \\ \dfrac{\partial^2 g}{\partial F_x\, \partial F_z} & \dfrac{\partial^2 g}{\partial F_y\, \partial F_z} & \dfrac{\partial^2 g}{\partial F_z^2} \end{bmatrix} \tag{5.8}$$

where

$$\frac{\partial^2 g}{\partial F_z^2} = 0 \qquad \frac{\partial^2 g}{\partial F_x\, \partial F_y} = -\frac{F_x F_y}{\left(F_x^2 + F_y^2\right)^{3/2}}$$

$$\frac{\partial^2 g}{\partial F_x\, \partial F_z} = 0 \qquad \frac{\partial^2 g}{\partial F_y^2} = \frac{F_x^2}{\left(F_x^2 + F_y^2\right)^{3/2}}$$

$$\frac{\partial^2 g}{\partial F_y\, \partial F_z} = 0 \qquad \frac{\partial^2 g}{\partial F_x^2} = \frac{F_y^2}{\left(F_x^2 + F_y^2\right)^{3/2}}$$

It is easily shown that all primary determinants of (5.8) are either positive or zero:

$$\frac{\partial^2 g}{\partial F_x^2} \geq 0$$

$$\frac{\partial^2 g}{\partial F_x^2}\,\frac{\partial^2 g}{\partial F_y^2} - \left[\frac{F_x F_y}{\left(F_x^2 + F_y^2\right)^{3/2}}\right]^2 = 0$$

$$\det(H_g) = 0$$

Therefore H_g is positive semidefinite which means that function g and the region it defines are convex. This result is coordinate independent since transformations are linear mappings. The intersection of this region with the convex region of linear constraints also yields a convex region.

Thus, the set of constraint expressions defines a convex set of solutions, which can be seen by simple inspection also to be closed and bounded (recall that the vertical components of the reaction forces on each vehicle are restricted by the forces exerted on the vehicle by the manipulator and its weight). The fact that the region inside the cone is only simply convex is not a problem because solutions are sought along a cord of a circle formed by the intersection of the friction cone with a horizontal plane. To illustrate this, consider the simple case of a mobile platform k with only three wheels.

Recall that $(F_z^{w_j})_k$ are restricted in terms of the interaction forces and the weight acting on the platform as in Eq. (5.1). If all kinematic quantities and load conditions are prescribed for the mechanism, then by using the two equations in the set (5.6) that illustrate the angular momentum preservation along axes x and y and Eq. (5.1), one can calculate analytically the three vertical components $(F_z^{w_j})_k$, $j = 1, \ldots, 3$. For each value of $(F_z^{w_j})_k$, a horizontal plane that intersects the corresponding friction cone along a closed curve is defined for each wheel. This closed curve is a circle because the axis of the cone is always vertical to the contact surface (in this case, the ground). Admissible solutions for the reaction forces from the ground should lie inside the cone and reach the horizontal plane defined by the solution for $(F_z^{w_j})_k$ (Fig. 4).

If we suppose that $(F_x^{w_j})_k$ can be prescribed by the applied torque at each wheel, then another plane, vertical this time, is defined for each wheel. The intersection of this plane with the circular disk inside the friction cone is a chord (Fig. 4). The endpoints of this chord correspond to the maximum centripetal forces that the wheel can exert. These limits can always be calculated through a condition for maximizing $\|(F_x^{w_j})_k\|$, since they are defined as the intersection of a (strictly) convex curve and a straight line.

The third component, $(F_y^{w_j})_k$, cannot be uniquely determined: there is one equation left out of (5.6), together with (5.5). Hence, the solutions for $(F_y^{w_j})_k$ belong to a one-dimensional space. Given the kinematic quantities of the mechanism, one can verify that whether the magnitude necessary for the prescribed motion centripetal force, $\|F_y\|$, can be provided by each wheel, i.e., if

$$\sum_j^w \left\|(F_y^{w_j})_k\right\| \geq \|F_y\|$$

6. CONCLUSION

In this work, a multiple mobile manipulator system handling a common deformable object is modeled. The approach introduces novelties in terms of systematic analysis and generality. It integrates many significant results on the fields of mobile robots, manipulators, and mechanical engineering. The model is based on Kane's methodology, which provides ease of construction of the equations, simplicity, physical insight, speed in simulation, and, finally, a control orientation. The approach allows for the manipulation of any deformable object of arbitrary shape using elastodynamic equations, and gives a lower limit for the complexity of the object's finite elements model so that control is implementable. Specifically, the simplest approximating grid structure for the object model is indicated. Within this context, the notion of operational space is generalized to include the case of a deformable object with arbitrary degrees of freedom. A global operational point is defined that enables the aplication of the augmented object approach for merging the mobile manipulators dynamic equations with the object model into a single set of dynamic equations. Then the closed chains formed are investigated. Why their treatment of Kane's framework is equivalent to conventional approaches is explained.

Nonholonomic constraints that are imposed on the system are included. It is pointed out that conventional velocity constraint equations are not sufficient to guarantee nonholonomic motion for a real system. A set of additional conditions is formed that can ensure mechanical stability and nonholonomic motion by regulation of interaction forces between the vehicles and the attached manipulators. The set of admissible solutions for the ground reaction forces is identified and a procedure is described for veryifying that contact stability is guaranteed and constraints are being respected. Emphasis is placed on these interaction forces and care is taken for them to become evident within the dynamic equations.

It is the authors' belief that, in this work, the problem of modeling multiple mobile manipulator systems that handle a deformable object is treated systematically with adequate generality, maintaining a clear insight on the physics of the problem. Hopefully, this framework will provide a safe and suitable groundwork for the development of control strategies for multiple mobile manipulator systems.

APPENDIX

For the sake of the text being self-contained, the main ideas underlying Kane's dynamic equation[18] are briefly introduced in this section. The equations are expressed in terms of generalized coordinates, generalized speeds, and generalized forces. Partial velocities and partial angular velocities are introduced and utilized in the derivation of each force contribution to the equations.

Consider a system with n degrees of freedom. One can define n generalized coordinates

$$\mathbf{q} = [q_1, \ldots, q_n]^T.$$

Generalized speeds are defined as

$$u_r = \sum_{s=1}^{n} \mathbf{Y}_{rs}\dot{\mathbf{q}}_s + \mathbf{Z}_r \qquad r = 1, \ldots, n$$

$\mathbf{Y}_{rs}$ and $\mathbf{Z}_r$ are functions of the generalized coordinates and time. These functions must be chosen so that the equations can be solved uniquely for $\dot{\mathbf{q}}_1, \ldots, \dot{\mathbf{q}}_n$. Use of the above definition for a specific mechanical system can yield simplified expressions of the systems equations.[18]

Suppose the system consists of m rigid bodies. The partial velocity and partial angular velocity of a point Q of a rigid body are defined as

$$\mathbf{v}_r^Q \stackrel{\Delta}{=} \frac{\partial \mathbf{v}^Q}{\partial u_r}, \qquad \boldsymbol{\omega}_r^Q \stackrel{\Delta}{=} \frac{\partial \boldsymbol{\omega}^Q}{\partial u_r}$$

where $\mathbf{v}^Q$ and $\boldsymbol{\omega}^Q$ are the velocity and angular velocity of point Q. The partial velocities can usually be derived by inspection, since the velocities are simply related to generalized speeds. Appropriate choice of generalized speeds can render this procedure trivial.

Suppose that $n-p$ nonholonomic constraints are imposed on the system. Using the nonholonomic constraints to express some generalized speeds in terms of the rest and then substituting for them, one obtains velocity expression with a reduced number of generalized speeds. The partial velocities calculated from these expressions are called nonholonomic and are given by

$$\tilde{\mathbf{v}}^Q = \frac{\partial \mathbf{v}^Q}{\partial u_r} \qquad \tilde{\boldsymbol{\omega}}_r^Q = \frac{\partial \boldsymbol{\omega}^Q}{\partial u_r} \qquad r = 1, \ldots, p$$

The generalized forces calculated using the nonholonomic partial velocities are also called nonholonomic.

In Kane's framework, generalized forces are classified as either *active* or *inertial*. Active forces are generally caused by the action of forces and torques external w.r.t. the system, such as contact, body, or field forces. On the other hand, as can be assumed by the name, inertial forces are the result of the inertia of the components of the system.

Suppose now that the set of contact and/or body forces and torques acting on a rigid body B is equivalent to a force $\mathbf{F}^B$ and a torque $\mathbf{T}^B$ acting on point Q of body B. The contribution of these forces to the set of nonholonomic generalized active forces $\tilde{F}_r$ is

$$\left(\tilde{F}_r\right)_B = \tilde{\mathbf{v}}^Q \cdot F^B + \tilde{\boldsymbol{\omega}}^B \cdot T^B \qquad r = 1, \ldots, p$$

It is known the inertial forces on B can be expressed as[18]

$$F^{*B} = -m^B \mathbf{a}^{cB}$$

where m^B is the mass of the rigid body on which the inertial force is acting and $\mathbf{a}^{cB}$ is the acceleration of the mass center, cB, of the body B. Inertial torques are expressed as[18]

$$T^{*B} = -\boldsymbol{\alpha}^B \cdot \mathbf{I}^B - \boldsymbol{\omega}^B \times \mathbf{I}^B \cdot \boldsymbol{\omega}^B$$

where $\boldsymbol{\alpha}^B$ and $\boldsymbol{\omega}^B$ are the angular acceleration and the angular velocity of body B, respectively. $\mathbf{I}^B$ is the central inertia dyadic of B.[18] The use of the dyadic enables the inertia matrix to be expressed in a coordinate independent way. A dyadic is a juxtaposition of vectors in a way that any two vectors expressed as

$$\mathbf{v} = \mathbf{w} \cdot \mathbf{ab} + \mathbf{w} \cdot \mathbf{cd} + \cdots$$
$$\mathbf{u} = \mathbf{ab} \cdot \mathbf{w} + \mathbf{cd} \cdot \mathbf{w} + \cdots$$

can be written in the form

$$\mathbf{v} = \mathbf{w} \cdot \mathbf{Q}$$
$$\mathbf{u} = \mathbf{Q} \cdot \mathbf{w}$$

where $\mathbf{Q}$ is the dyadic:

$$\mathbf{Q} = \mathbf{ab} + \mathbf{cd} + \cdots$$

The contribution of the inertial forces acting on B to the generalized inertial forces for the system is

$$\left(\tilde{F}_r^*\right)_B = \tilde{\mathbf{v}}_r^Q \cdot F^{*B} + \tilde{\boldsymbol{\omega}}_r^Q \cdot T^{*B}$$

Kane's dynamic equations can be summarized in the following relation, which sums all contributions from the rigid bodies of the system:

$$\tilde{F}_r + \tilde{F}_r^* = 0$$

REFERENCES

1. E. Papadopoulos and S. Dubowsky, "Coordinated manipulator/spacecraft motion control for space robotic systems," *IEEE Int. Conf. Robotics and Automation*, 1991, pp. 1696–1701.
2. E. Papadopoulos and S. Dubowsky, "On the nature of control algorithms for free-floating space manipulators," *IEEE Trans. Robotics Automat.*, **7**(6), 750–758, 1991.
3. T. J. Tarn, G. A. Shoults, and S. P. Yang, "Dynamic model for an underwater underwater vehicle with multiple robotic manipulators," *Pre-Proc. 6th Int. Advanced Robotics Program*, Toulon-La Seyne, March 1996, pp. 1–23.
4. H. Mahesh, J. Yuh, and R. Lakshmi, "A coordinated control of and underwater vehicle and robotic manipulator," *J. Robotic Sys.*, **8**(3), 339–370, 1991.
5. Y. Yamamoto, "Control and coordination of locomotion and manipulation of a wheeled mobile manipulator" Ph.D. thesis, Dept. of Computer and Information Science, School of Engineering and Applied Science, University of Pennsylvania, Philadelphia, PA, August 1994.
6. Y. Yamamoto and X. Yun, "Coordinating locomotion and manipulation of a mobile manipulator," *IEEE Trans. Automatic Control*, **39**(6), 1326–1332, 1994.
7. O. Khatib, K. Yokoi, D. Ruspini, R. Holmberg, A. Casal, and A. Baader, "Force strategies for cooperative tasks in multiple mobile manipulation systems," *Int. Symp. on Robotic Research*, Munich, October 1995.
8. S. Dubowsky and E. E. Vance, "Planning mobile manipulator motions considering vehicle dynamic stability constraints," *Proc. IEEE Int. Conf. on Robotics and Automation*, 1989, pp. 1271–1276.
9. M. W. Chen and A. M. S. Zalzala, "Dynamic modelling and genetic-based motion planning of mobile manipulator systems with nonholonomic constraints," Research Report 600, Robotics Research Group, Dept. of Automatic Control and Systems Engineering, University of Sheffield, UK, September 1995.
10. N. A. M. Hootsmans, "The motion control of manipulators on mobile vehicles," Ph.D. thesis, Dept. of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA, August 1994.
11. Y. Yamamoto and X. Yun, "Effect of the dynamic interaction on coordinated control of mobile manipulators," *IEEE Trans. Robotics Automat.*, **12**(3), 816–824, 1996.
12. Ju. I. Neimark and F. A. Fufaev, *Dynamics of Nonholonomic Systems*, Translations of Mathematical Monographs, Vol. 33, American Mathematical Society, Providence, RI, 1972.
13. R. Brockett, *Control Theory and Singular Riemannian Geometry*, New Directions in Applied Mathematics, Springer, Berlin, 1981.
14. I. Kolmanovsky and N. H. McClamroch, "Developments in nonholonomic control problems," *IEEE Control Sys.*, **15**(6), 20–36, 1995.
15. B. d'Andrea-Novel, G. Campion, and G. Bastin, "Modelling and state feedback control of nonholonomic mechanical systems," *Proc. 1991 IEEE Conf. on Decision and Control*, December 1991.
16. G. J. Pappas and K. J. Kyriakopoulos, "Modeling and feedback control of nonholonomic mobile vehicles," *Proc. 1992 IEEE Conf. on Decision and Control*, December 1992.
17. A. Isidori, *Nonlinear Control Systems*, Springer-Verlag, Berlin, 1989.
18. T. R. Kane and D. A. Levinson, *Dynamics: Theory and Applications*, McGraw-Hill, New York, 1985.
19. O. Khatib, "A unified approach for motion and force control of robot manipulators: The operational space formulation," *IEEE J. Robotics Automat.*, **RA-3**(1), 43–53, 1987.
20. K. Thanjavur and R. Rajogopalan, "Ease of dynamic modelling of wheeled mobile robots (WMRs) using Kane's approach," *IEEE Int. Conf. Robotics and Automation*, Albuquerque, New Mexico, 1997, pp. 2926–2931.
21. T. R. Kane and D. A. Levinson, "The use of Kane's dynamical equations in robotics," *Int. J. Robotics Res.*, **2**(3), 3–21, 1986.
22. O. Khatib, "Inertial properties in robotic manipulation: An object-level framework," *Int. J. Robotics Res.*, **14**(1), 19–36, 1995.
23. W.-H. Zhu, Y.-G. Xi, Z.-J. Zhang, Z. Bien, and J. De Schutter "Virtual decomposition based control for generalized high dimensional robotic systems with complicated structure," *IEEE Trans. Robotics Automat.*, **13**(3), 411–436, 1997.
24. T. J. Tarn, A. K. Bejczy, and X. Xun, "Design of dynamic control of two cooperating robot arms: Closed chain formulation," *IEEE Int. Conf. Robotics and Automation*, Raleigh, 1987, pp. 7–13.
25. Y. Nakamura and M. Ghoudoussi, "Dynamics computation of closed-link robot mechanisms with nonredundant and redundant actuators," *IEEE Trans. Robotics Automat.*, **5**(3), 294–302, 1989.
26. J. Y. S. Luh and Y.-F. Zheng, "Computation of input generalized forces for robots with closed kinematic chain mechanisms," *IEEE J. Robotics Automat.*, **RA-1**(2), 95–103, 1985.
27. S.-K. Lin, "Dynamics of the manipulator with closed chains," *IEEE Trans. Robotics Automat.*, **6**(4), 496–501, 1990.
28. D. A. Smith, "Reaction force analysis in generalized machine systems," *ASME J. Eng. Industry*, **95**(2), 617–623, 1973.
29. R. Featherstone, *Robot Dynamics Algorithms*, Kluwer, Dordrecht, 1987.
30. L. T. Wang and M. J. Kuo, "Dynamic load-carrying capacity and inverse dynamics of multiple cooperating robotic manipulators," *IEEE Trans. Robotics Automat.*, **10**(1), 71–77, 1994.
31. Y. Yamamoto, H. Eda, and X. Yun, "Coordinated task execution of a human and a mobile manipulator,"

IEEE Int. Conf. on Robotics and Automation, Minneapolis, MN, April 1996.
32. L. Sciavicco and B. Siciliano, *Modeling and Control of Robot Manipulators*, McGraw-Hill, New York, 1996.
33. J. J. Craig, *Introduction to Robotics, Mechanics, and Control*, Addison-Wesley, Reading, MA, 1989.
34. D. Williams and O. Khatib, ''The virtual linkage: A model for internal forces in multi-grasp manipulation,'' *Proc. IEEE Int. Conf. on Robotics and Automation*, 1993, pp. 1025–1030.
35. Z. Vafa and S. Dubowsky, ''The kinematics and dynamics of space manipulators: The virtual manipulator approach,'' *Int. J. Robot. Res.*, **9**(4), 3–21, 1990.
36. Y. Nakamura, *Advanced Robotics: Redundancy and Optimization*, Addison-Wesley, Reading, MA, 1991.