Hierarchical Control via Approximate Simulation and Feedback Linearization

Jie Fu, Shridhar Shah and Herbert G. Tanner

***Abstract*— This paper considers the problem of hierarchical trajectory planning and control for a class of nonlinear systems which are feedback linearizable. The proposed hierarchy builds on the notion of approximate simulation relations. We use the diffeomorphic transformation between the feedback linearized system and the nonlinear system, along with the associated interface between the abstraction and the linearization of the concrete system obtained via feedback, to recast the problem of trajectory planning and control for the nonlinear system into a reduced dimensional space. The formal abstraction framework then enables the control to be mapped to the space of the concrete nonlinear system using the feedback linearized system as a "bridge". Two different hierarchical control architectures are proposed for this framework and the method is demonstrated with a quadrotor system.**

Keywords: hierarchical control, abstraction, differential flatness, approximate simulation.

I. INTRODUCTION

Advances in manufacturing techniques have allowed one to build miniaturized robotic systems; however these systems can carry only a small payload and possess limited computation resources. Implementing sophisticated control and planning solutions on these systems can be difficult. In addition, miniaturization sometimes leads to new geometries and motion generation mechanisms, which if modeled accurately yield high dimensional dynamics. The design of controllers for complex, high dimensional nonlinear systems has always been challenging. In such cases, a hierarchical approach to design, which includes a detailed complex model at a lower level and a simpler coarse model at the upper level may be preferable. Control can be done in the coarse model and then the solution can be implemented on the complex system. The challenge here is to ensure consistency between the coarse and the detailed representations so that some guarantees of implementability and performance can be ensured.

Different hierarchical approaches have been proposed for continuous systems in [1]–[3]. The notion of hierarchical control framework for continuous control systems proposed in [1] utilizes vector fields related through a smooth map to obtain abstract systems. The method is illustrated with linear systems. An extension of the abstraction to nonlinear (affine) systems is presented in [4], where it is shown that the local accessibility of the abstract system is equivalent to the local accessibility of the nonlinear one. In [4], the abstraction map is constructed for the purpose of formal verification.

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A constructive method for hierarchical trajectory planning for nonlinear affine systems is proposed in [2], and ensures that the trajectories generated in the abstract system are feasible in the concrete one. The notion of simulation relations [5] is employed for hierarchical control design in [3], while a relaxation where the simulation relation is approximate is proposed in [3]. The control law for the abstracted system is translated from the abstract to the concrete system through an interface, and a constructive procedure is offered for the case of linear controllable systems. The important aspect of the approximate simulation framework is that the output of the abstract system is not required to match the output of the concrete system exactly. Instead, the difference between their output trajectories is bounded to an adjustable tolerance.

The hierarchical trajectory generation method proposed in [2] is similar to the one suggested herein, although here abstract trajectories uniquely define the output trajectories of the concrete system. The emphasis in [2] is on trajectory generation rather than control design, while the hierarchical control design of [3] gives constructive tools only for linear controllable systems. In this paper, we extend these methods so that they also apply to a class of nonlinear systems which can be feedback linearized (possibly after prolongations) by linking differential flatness theory with the framework of [3]. Intuitively, feedback linearization maps the original nonlinear system into a linear one in Brunovsky canonical form. The benefit is that at that state the system representation is linear and the results of [3] apply directly.

We present two methods to construct controllers hierarchically and show that differentially flat systems admit linear abstractions where the number of state variables equals the number of inputs in the system. In one of the hierarchical methods, the linearized system is used explicitly in the control loop to construct a dual feedback controller robust to initial condition errors or disturbances. In this way, one can apply the methodology to systems for which the set of flat outputs can only be obtained after reasonable approximations of the system dynamics. The second one uses a simpler hierarchical control architecture more preferable to differentially flat nonlinear dynamics in no need for such approximation.

We demonstrate the application of our method through an example of quadrotor system. It should be noted that the feedback linearization is a common method for control of quadrotors [6], [7]. However, the presented paper goes one step further and reduces the dimension of the system through approximate simulation relation. The contribution of this paper is to reduce complexity and dimensionality of nonlinear systems that are feedback linearizable through approximate simulation relations.

Section II which follows reviews approximate simulation relations from [3] and briefly comments on the differential flatness property of dynamic systems. Section III shows how feedback linearizable nonlinear systems are abstracted and illustrates the design of abstraction based hierarchical controllers. Section IV presents the design and simulation results for the dynamical model of a quadrotor helicopter, and the paper concludes with Section V.

II. PRELIMINARIES

A. Abstractions of linear systems

Consider the following two linear systems:

$$\Sigma_1 : \quad \dot{\xi} = A\xi + Bu, \quad y = C\xi \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the concrete state, $u \in \mathbb{R}^m$ is the concrete control input, $y \in \mathbb{R}^m$ is the concrete output.[1] It is also assumed that (A, B) is stabilizable, i.e., there exists a $m \times n$ matrix K for which $A + BK$ is Hurwitz. Consider also

$$\Sigma_2 : \quad \dot{z} = Fz + Gv, \quad \eta = Hz \tag{2}$$

where $z \in \mathbb{R}^p$ is the abstract state, $v \in \mathbb{R}^k$ is the abstract input and $\eta \in \mathbb{R}^m$ is the abstract output. It is assumed $\mathrm{rank}(B) = m$ and $\mathrm{rank}(C) = m$ and $p \leq n$ as Σ_2 is ideally smaller in dimension compared to Σ_1.

System Σ_2 is Π-related [3] to Σ_1 if there is a $p \times n$ matrix Π such that for all $\xi \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ there exists $v \in \mathbb{R}^k$ that satisfies $\Pi(A\xi + Bu) = F\Pi\xi + Gv$ and $C = H\Pi$.

If Σ_1 and Σ_2 are Π-related, then for any state trajectory $\xi(t)$ there exists a state trajectory $z(t)$ such that for all t, $z(t) = \Pi\xi(t)$ and the output trajectory satisfies $y(t) = \eta(t)$ [1]. Given Σ_1, we can construct Σ_2 to be Π-related to Σ_1 and call it an *abstraction* of Σ_1.

Thinking of P as the pseudo-inverse of Π, we have the following [3, Theorem 4]:

Theorem 1 ([3]): P is an injective map such that

$$\mathrm{im}(AP) \subseteq \mathrm{im}(B) + \mathrm{im}(P) \text{ and } \mathrm{im}(P) + \ker(C) = \mathbb{R}^n.$$

Let D, E and Π to be chosen such that $\mathrm{im}(P) \oplus \mathrm{im}(D) = \mathbb{R}^n$ ($\oplus$ denotes direct sum), $\mathrm{im}(D) \subseteq \ker(C)$ and $P\Pi + DE = I_n$, where I_n is a $n \times n$ identity matrix. Let F and Q be chosen such that $AP = PF - BQ$, and set $H = CP$ and $G = \begin{bmatrix} \Pi B & \Pi AD \end{bmatrix}$. Then Σ_2 is Π-related to Σ_1.

Through an *interface* between Σ_1 and Σ_2, which is a function of v, ξ, z, the concrete input u can be obtained in a way that the difference between the output trajectories of Σ_1 and Σ_2 is bounded from above for all t. The bound on the difference between the output trajectories is given by a *simulation function* $\mathcal{V}(z, \xi)$ [3, Lemma 1].

Lemma 1 ([3]): There exists a positive definite matrix M and a positive scalar number λ such that (for a stabilizing K of Σ_1) the following matrix inequalities hold:

$$\begin{aligned} &M \geq C^T C \\ &(A + BK)^T M + M(A + BK) \leq -2\lambda M \end{aligned} \tag{3}$$

[1] In [3], one may have $y \in \mathbb{R}^s$ where $s \neq m$.

The matrix M in (3) is related to the matrix P of Theorem 1, and together they are used to define the simulation function and the system's interface:

Theorem 2 ([3]): Let us assume there exists a $n \times p$ matrix P and an $m \times p$ matrix Q such that

$$PF = AP + BQ \text{ and } H = CP \ .$$

Then the function $\mathcal{V}(z, \xi) = \sqrt{(Pz - \xi)^T M (Pz - \xi)}$ is a simulation function of Σ_2 by Σ_1 and an associated interface is given by

$$u_v(v, z, \xi) = Rv + Qz + K(\xi - Pz) \tag{4}$$

where R can be an arbitrary $m \times k$ matrix.

If one defines [2]

$$\gamma(v) = \frac{\|\sqrt{M}(BR - PG)\|}{\lambda} v \tag{5}$$

(a $\mathcal{K}$-class function) then for all $v \in \mathbb{R}^k$ it holds that $\gamma(\|v\|) < \mathcal{V}(z, \xi)$ [3]. The function γ defined by (5) is minimal for

$$R = (B^T M B)^{-1} B^T M P G \ . \tag{6}$$

For Σ_2 steered by $v(t)$ and Σ_1 by $u_v(v(t), z(t), \xi(t))$ we will have [3]

$$\|y(t) - \eta(t)\| \leq \max\{\mathcal{V}(z(0), \xi(0)), \gamma(\|v\|_\infty)\} \ . \tag{7}$$

B. Differential flatness

We extend the results of the previous section to a class of nonlinear systems that enjoy a structural property known as differential flatness. For a more comprehensive introduction to flatness the reader is referred to [8].

Consider the multiple-input-multiple-output (MIMO) nonlinear system

$$\Sigma_0 : \begin{cases} \dot{x}(t) = f(x) + \sum_{j=1}^m g_j(x) w_j \\ y_i = h_i(x), \quad i = 1, 2, \ldots, m \end{cases} \tag{8}$$

where $x \in \mathbb{R}^n$, $w_j \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$.

The system is static feedback linearizable (SFL) if the sum of relative degrees r_i of outputs y_i satisfy $r \triangleq \sum_{i=1}^m r_i = n$. In the case that a system is not SFL, it may be dynamic feedback linearizable (DFL) through prolongation. In this paper we restrict our focus to the type of nonlinear systems that can be feedback linearized either through SFL or DFL. Assume that (8) is either SFL or DFL (and prolonged if necessary). Then there exists a diffeomorphism $\xi = T(x)$ that transforms Σ_0 into a linear controllable system with state

$$\xi = (y_1, \ldots, y_1^{(r_1-1)}, y_2, \ldots, y_2^{(r_2-1)}, \ldots, y_m, \ldots, y_m^{(r_m-1)})^T$$

[2] Since M is positive definite by Theorem 2, it is always nonsingular and thus $\sqrt{M}$ exists.

where $y_i^{(r_j)}$ denotes the r_j-order time derivative of y_i:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 = L_f^1 h_1(x), \quad \xi_1 \triangleq y_1. \\
&\vdots \\
\dot{\xi}_{r_1} &= L_f^{r_i} h_1(x) + \sum_{j=1}^{m} L_{g_j}(L_f^{r_1-1} h_1) w_j \triangleq u_1 \\
\dot{\xi}_{r_1+1} &= L_f^1 y_2, \quad \xi_{r_1+1} \triangleq y_2 \\
&\vdots \\
\dot{\xi}_r &= L_f^{r_m} h_m(x) + \sum_{j=1}^{m} L_{g_j}(L_f^{r_m-1} h_m) w_j \triangleq u_m \quad .
\end{aligned} \tag{9}$$

In the above L stands for the Lie derivative.

When the diffeomorphic transformation $T(x)$ can be found to bring (8) into the the Brunovsky canonical form (9), we say that Σ_0 is *differentially flat* [8]. In such a case, each of the states and the inputs of Σ_0 is a function of y and a finite number of its (higher order) time derivatives. The components of y are referred to as *flat outputs*.

III. HIERARCHICAL DESIGN

In this section we demonstrate how an abstraction of a feedback linearized system in Brunovksy canonical form (9) can be related to the nonlinear system Σ_0. We saw that a differentially flat nonlinear system Σ_0 can take the form of the concrete linear system given by (1):

$$\Sigma_1: \quad \dot{\xi} = A\xi + Bu, \quad y = h(x) = C\xi \tag{10}$$

and the m-dimensional input of Σ_0 and Σ_1 can be related through (9); written compactly, the inputs are related as

$$w = a(x) + b(x)u$$

where $a \in \mathbb{R}^m$, $b \in \mathbb{R}^{m \times m}$, and the elements of both a and b are (generally nonlinear) functions of x.

A. Features of abstract system

Theorem 3: Every (statically or dynamically) feedback linearizable system Σ_0 admits a linear abstraction Σ_2 in which the number of states equals the number of inputs.

Proof: Consider the single-input-single-output (SISO) case where the A, B, C matrices in Σ_1 are of the form

$$A = J_{0,r}, \quad B = \begin{bmatrix} 0_{1,r-1} & 1 \end{bmatrix}^T, \quad C = \begin{bmatrix} 1 & 0_{1,r-1} \end{bmatrix}$$

where r is the relative degree of the flat output y. The Jordan block $J_{0,r}$ has dimension $r \times r$ block and eigenvalue 0, I_k is the identity in $\mathbb{R}^{k \times k}$, $0_{k_1,k_2}$ is the zero in $\mathbb{R}^{k_1 \times k_2}$, $0_k \equiv 0_{k,k}$.

Given that (C, A) is observable, P can be chosen as a matrix whose range spans an arbitrary A-invariant subspace of dimension greater than or equal to m [3], where m is the number of outputs in Σ_1. (In the SISO case, $m = 1$.) Choose $P^T = \begin{bmatrix} 1 & 0_{1,r-1} \end{bmatrix}$ and verify that $\mathrm{im}(P) \oplus \ker(C) = \mathbb{R}^r$. According to Theorem 1, the choice of $D = \begin{bmatrix} 0_{1,r-1} \\ I_{r-1} \end{bmatrix}$, $\Pi = \begin{bmatrix} 1 & 0_{1,r-1} \end{bmatrix}$, together with $F = 0$, $Q = 0$ so that $AP = PF - BQ$, yield $H = CP = 1$ and $G = \begin{bmatrix} \Pi B & \Pi A D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0_{1,r-2} \end{bmatrix}$. (Redefining the dimension of the inputs of the abstract system, one can simply take $G = 1$.) After substituting in (2), the SISO Σ_2 becomes a single dimensional controllable linear system

$$\dot{z} = Fz + Gv = v, \quad \eta = z \quad .$$

Consider now the MIMO case. For each flat output $y_i, i = 1, \ldots, m$, the dynamics of the associated state variables $\xi_{i1}, \ldots, \xi_{ir_i}$ can each be described by a SISO subsystem. The abstract system is essentially a collection of single integrators, one for each flat output y_i. Once the variables are stacked, the abstract system takes the (trivial) form

$$\dot{z} = v, \quad \eta = z \tag{11}$$

where $z, v \in \mathbb{R}^m$. ■

B. Hierarchical Planning and Control Design

The objective is to develop a feedback controller for Σ_0 that steers it from an initial state $x(t_0)$ to a final state $x(t_f)$, with t_0, t_f known. Given that y and its higher order derivatives are one-to-one mapped to ξ (and x through the diffeomorphism T), planning a trajectory from $x(t_0)$ to $x(t_f)$ is reduced to finding a trajectory in the (flat) output space y that satisfies specific initial and terminal constraints.

We present two different control architectures depending on the fidelity of the abstractions obtained through feedback linearization. Both architectures can compensate for errors (mismatches) in the initial condition $x(t_0)$ of Σ_0, and the difference lies in how errors induced by modeling simplification in the nonlinear dynamics, which are required to find the flat outputs, are compensated for.

In the first case we assume that flat outputs can be found without simplification of the dynamics of Σ_0. Then the feedback linearized system Σ_1 is an equivalent representation of the original system. We can plan trajectories that link $x(t_0)$ to $x(t_f)$ and implement them exactly. In the second case, we essentially assume that flat outputs can only be found if certain simplifications have to be made in the dynamics of nonlinear system Σ_0. The simplifications would inevitably introduce a mismatch between the image ξ_d of the states x of the nonlinear system under T, and the states ξ_d of the feedback linearization Σ_1^d of the simplified nonlinear dynamics $\tilde{\Sigma}_0$. Then, an additional feedback loop is introduced to compensate for the mismatch.

1) When flat outputs are acquired without simplification: If Σ_0 is feedback linearizable, there is a diffeomorphic transformation $\xi = T(x)$, under which the initial and final conditions $x(t_0)$, $x(t_f)$ are mapped to some initial and final conditions $\xi(t_0) = T(x(t_0))$, $\xi(t_f) = T(x(t_f))$.

For given initial $\xi(t_0)$ and desired final states $\xi(t_f)$ for Σ_1, the corresponding initial and final states of its abstraction Σ_2 are given as $z(t_0) = \Pi\xi(t_0)$ and $z(t_f) = \Pi\xi(t_f)$. Since Σ_2 of (11) is fully actuated, the design of control input v to produce trajectories satisfying $z(t_0) = \Pi\xi(t_0)$ and $z(t_f) = \Pi\xi(t_f)$ is straightforward. With v specified, the control input u of Σ_1 is given by the associated interface u_v defined in (4), with R given in (6) to guarantee minimal deviation of y

from η (see (7)). Note, however, that in our formulation we have $\eta = z$ so the simulation function practically measures the difference between the output of the concrete (feedback) linearization Σ_1, and the abstract state of Σ_2.

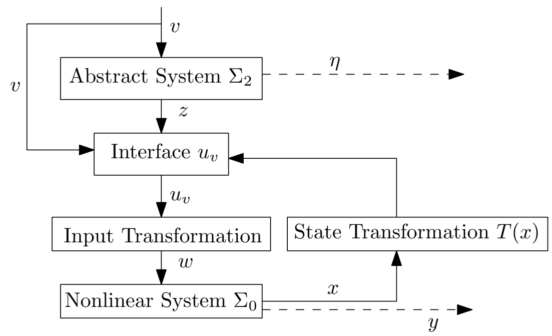

Fig. 1. A two-level hierarchical control system architecture. The scheme enables planning and control using models of smaller dimension when the concrete nonlinear model Σ_0 is differentially flat.

This architecture is depicted in Fig. 1. Given Π is surjective, not all the components of $x(t_0)$ and $x(t_f)$ (read $\xi(t_0)$ and $\xi(t_f)$) can be matched when regulating $z(t)$ through v. Some of this information is lost in the abstraction process. However, even if the abstraction Σ_2 is initiated from an arbitrary initial condition, the approximate simulation relation [3] that has been established between Σ_1 and Σ_2 guarantees that at least m of these components will approximately come and remain close to the desired values in $x(t_f)$.

2) When flat outputs are acquired after simplifications: It is often the case that no appropriate output vector y can be found for the system to be feedback linearized. In certain cases, some reasonable simplifications in modeling system dynamics may enable us to obtain the set of flat outputs for (still nonlinear) system $\tilde{\Sigma}_0$. Due to the errors introduced by the simplifications, an additional control loop may be necessary to generate corrective control action in Σ_0 that compensates for the discrepancy between Σ_0 and Σ_1.

The augmented architecture (in Fig. 2) contains an additional dynamical system — the *flat reference system* Σ_1^d:

$$\Sigma_1^d : \quad \dot{\xi}_d = A\xi_d + Bu_v, \quad \eta_d = y_d \ .$$

This system is the (dynamic) feedback linearization of the *simplified* system $\tilde{\Sigma}_0$. Implementing the simulation interface input u_v on Σ_1^d generates a reference trajectory $\xi_d(t)$ on the flat space. If Σ_0 were flat (no simplifications) then $\xi_d(t)$ would be equal to $T(x(t))$ with $x(t)$ being the trajectory of Σ_0 driven by input $a(x) + b(x)u_v$. But with $x(t)$ being the state of Σ_0 (not $\tilde{\Sigma}_0$), $\xi = T(x)$ and ξ_d do not match. This is where the additional control loop is closed around.

Let $y_d \triangleq C\xi_d$ and $y = h(x) = CT(x)$ (see (10)). Define $e \triangleq y - y_d$ and construct a *pseudo-input* $u^F = (u_1^F, u_2^F, \ldots, u_m^F)^T$ for Σ_1 component-wise as

$$u_i^F = y_{i\,d}^{(r_i)} - k_{i\,1} e_i^{(r_i - 1)} - k_{i\,2} e_i^{(r_i - 2)} - \ldots - k_{i\,r_i} e_i \quad (12)$$

where $y_{i\,d}, \ldots, y_{i\,d}^{(r_i-1)}$ are obtained through ξ_d, and $y_{i\,d}^{(r_i)}$ equals $u_v(i)$. The gains $k_{i\,1}, \ldots, k_{i\,r_i}$ are chosen so that the polynomial $s^{r_i} + k_{i\,1}s^{r_i - 1} + \cdots + k_{i\,r_i}$ is Hurwitz. What u^F does is to modify the input for Σ_0 as $w = a(x) + b(x)u^F$ so that the mismatch between the intended concrete state $T^{-1}(\xi_d)$ of $\tilde{\Sigma}_0$ and the true state x of Σ_0 is minimized. This hierarchical control architecture is shown in Fig. 2.

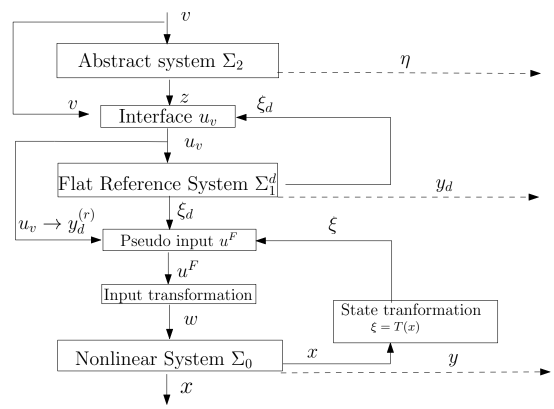

Fig. 2. A three-level hierarchical control system architecture. It applies to nonlinear systems that can be made differentially flat after some approximations. An additional control loop is established through u^F to mitigate the effect of the approximation.

The three-level hierarchical control tends to be more robust to disturbances, initial and terminal state errors compared to the one with two-levels. The two-level hierarchy can be applied to systems that can be feedback linearized after simplification, but its capability to suppress errors and disturbances through its own internal feedback loop is more limited. In the two-level hierarchy there appears to be an upper bound on λ in (5) (enters in (3)) and regulates the magnitude of the residual output trajectory error.

It is known that there may exist singularities in the controller derived through differential flatness theory [9]. Also there exists input constraints in all practical systems. To avoid singularities and ensure the feasibility of control inputs, a general approach is to optimize the output trajectory in flat space [10]. In the proposed hierarchical control system architecture, an optimization problem can be formulated for planning trajectory in the abstract system taking into consideration of input constraints and the singularities.

IV. Example

In this section we consider the dynamical model of a quadrotor [11]. A quadrotor is an underactuated flying vehicle with four inputs (thrust forces or angular velocities of each rotor). Schematically, such a system is shown in Fig. 3.

The equations of motions of a quadrotor robot after input transformation (see [11]) can be described as

$$\begin{aligned} \ddot{x} &= w_1(\cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi); \;\; \ddot{\theta} = w_2 L \\ \ddot{y} &= w_1(\sin\phi \sin\theta \cos\psi - \cos\phi \sin\psi); \;\; \ddot{\psi} = w_3 L \\ \ddot{z} &= w_1(\cos\theta \cos\psi) - g; \;\; \ddot{\phi} = w_4 \end{aligned} \quad (13)$$

where L (not to be confused with the Lie derivative) is the length from center to rotor and $w = (w_1, w_2, w_3, w_4)^T$ is the

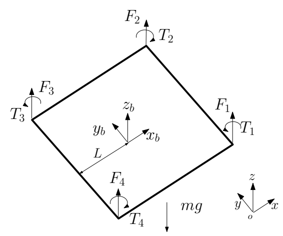

Fig. 3. A quadrotor helicopter configuration with roll(rotation along x-axis), pitch(rotation along y-axis), yaw(rotation along z-axis), where the Euler angles are ψ, θ, and ϕ respectively.

vector of inputs obtained through a linear transformation of the thrust forces F_i for $i = 1, \ldots, 4$. One can refer to [12] for details on the derivation of (13).

The state vector x for this nonlinear model is defined as

$$[x_1, x_2, \ldots, x_{12}] = [x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \dot{\phi}, \dot{\theta}, \dot{\psi}] \ .$$

For our numerical tests, $L = 3$. The desired initial and final conditions of the nonlinear system are

$$x(t_0) = \begin{bmatrix} 1\ 2\ 3\ 0\ 0.02\ 0.1\ 0.1\ 0.1\ 0.3\ 0.1\ 0.1\ 0.1\ 1\ 0.1 \end{bmatrix}^T$$
$$x(t_f) = \begin{bmatrix} 5\ 7\ 8\ 0\ 0.02\ 0\ 0.02\ 0.03\ 0.03\ 0\ 0\ 0.1\ 2\ 0.1 \end{bmatrix}^T \ .$$

Assuming that the yaw angle ϕ is sufficiently small yields a slightly more simplified model:

$$\tilde{\Sigma}_0 : \begin{cases} \ddot{x} = w_1(\sin\theta\cos\psi); & \ddot{\theta} = w_2 L \\ \ddot{y} = -w_1 \sin\psi; & \ddot{\psi} = w_3 L \\ \ddot{z} = w_1(\cos\theta\cos\psi) - g; & \ddot{\phi} = w_4 \end{cases}$$

It can be shown that $\tilde{\Sigma}_0$ is DFL after prolongation with $x_{13} = w_1$ and $x_{14} = \dot{w}_1$ [11]. The input for $\tilde{\Sigma}_0$ after prolongation is $w' = [\ddot{w}_1, w_2, w_3, w_4]^T$. The flat outputs are

$$y = [x_1, x_2, x_3, x_4]^T = [x, y, z, \phi]^T \ .$$

The vector relative degree of the system is $[r_1, r_2, r_3, r_4] = [4, 4, 4, 2]$. Under the small angle approximation for ϕ, its dynamics are decoupled from the rest of the system, and a PD controller can be independently designed [11] to set ϕ tracking a reference $\phi_d(t) = 0, \forall t \in [t_0, t_f]$, $w_4 = \ddot{\phi}_d - k_{\phi 1}(\dot{\phi} - \dot{\phi}_d) - k_{\phi 2}(\phi - \phi_d)$. (That ensures that the small angle assumption remains valid.) For the rest of the states we set $\xi = [x, \dot{x}, \ddot{x}, x^{(3)}, y, \dot{y}, \ddot{y}, y^{(3)}, z, \dot{z}, \ddot{z}, z^{(3)}]^T$ and form the feedback linearized system as

$$\Sigma_1 : \dot{\xi} = A\xi + Bu, \quad \eta = C\xi.$$

where $A = \mathrm{diag}(\mathrm{J}_{0,4}, \mathrm{J}_{0,4}, \mathrm{J}_{0,4})$. With $\xi = T(x)$ we get

$$\xi(t_0) = \begin{bmatrix} 1\ 0.1\ 0.02\ 0.10\ 2\ 0.1\ -0.09\ -0.11\ 3\ 0.3\ -8.82\ 0.09 \end{bmatrix}^T$$
$$\xi(t_f) = \begin{bmatrix} 5\ 0.02\ 0.04\ 0\ 7\ 0.03\ 0\ -0.2\ 8\ 0.03\ -7.81\ 0.1 \end{bmatrix}^T \ .$$

The input $w = [\ddot{w}_1, w_2, w_3]$ (of Σ_0) is obtained from input u (of Σ_1) through the nonlinear transformation (where s_φ, c_φ abbreviate $\sin\varphi$, $\cos\varphi$ respectively)

$$\begin{bmatrix} \ddot{w}_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \mathrm{s}_\theta & w_1 \mathrm{c}_\theta & 0 \\ -\mathrm{s}_\phi & 0 & -w_1 \mathrm{c}_\phi \\ \mathrm{c}_\theta\, \mathrm{c}_\phi & -w_1 \mathrm{s}_\theta\, \mathrm{c}_\phi & -w_1 \mathrm{c}_\theta\, \mathrm{s}_\phi \end{bmatrix}^{-1}$$
$$\cdot \begin{bmatrix} -2\dot{w}_1 \dot{\theta} \mathrm{c}_\theta + \mathrm{w}_1 \dot{\theta}^2 \mathrm{s}_\theta + \mathrm{u}(1) \\ 2\dot{w}_1 \dot{\phi}_\phi - \mathrm{w}_1 \dot{\phi}^2 \mathrm{s}_\phi + \mathrm{u}(2) \\ 2\dot{w}_1 \dot{\theta} \mathrm{s}_\theta\, \mathrm{c}_\phi + 2\dot{\mathrm{w}}_1 \dot{\phi} \mathrm{c}_\theta\, \mathrm{s}_\phi - 2\mathrm{w}_1 \dot{\theta}\dot{\phi} \mathrm{s}_\theta\, \mathrm{s}_\phi + \mathrm{w}_1 (\dot{\theta}^2 - \dot{\phi}^2) \mathrm{c}_\theta\, \mathrm{c}_\phi + \mathrm{u}(3) \end{bmatrix}$$

that enables the abstract system Σ_2 to be constructed by setting $F = 0$, $G = I_3$, $H = I_3$, $P^T = \mathrm{diag}(\mathrm{P}_4, \mathrm{P}_4, \mathrm{P}_4)$ for $P_k = \begin{bmatrix} 1 & 0_{1,k-1} \end{bmatrix}$, $\Pi = P^T$:

$$\Sigma_2 : \dot{z} = v, \quad r = \eta = z$$

where $v, z, r \in \mathbb{R}^3$. With $z = \Pi\xi$ we have

$$z(t_0) = \begin{bmatrix} 1\ 2\ 3 \end{bmatrix}^T, \quad z(t_f) = \begin{bmatrix} 5\ 7\ 8 \end{bmatrix}^T \ .$$

In the abstract system Σ_2, as it is fully actuated, we plan $z(t)$ to be a straight line from $z(t_0)$ to $z(t_f)$. This yields $v = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T = \begin{bmatrix} 0.2 & 0.25 & 0.25 \end{bmatrix}^T$.

A. Two-level hierarchical control design

Ignoring for now the errors introduced by the small angle assumption, the two-level hierarchical control design approach is adopted. Setting $\lambda = 3$ we have $R = 119.868\, I_3$, $Q = \mathbf{0}_3$ and $K = \mathrm{diag}(K_b, K_b, K_b)$, where $K_b = 10^3 \times [-3.129, -1.678, -0.3294, -0.025]$ and $\begin{bmatrix} k_{\phi 2} & k_{\phi 1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$. We assume that the actual initial state of Σ_0, $x_e(t_0)$ does not coincide with $x(t_0)$, but is rather equal to

$$x_e(t_0) = \begin{bmatrix} 1.5\ 2.5\ 3.5\ 0\ 0.02\ 0\ 0.01\ 0.01\ 0.03\ 0\ 0\ 0.1\ 10\ 0.1 \end{bmatrix}^T \ .$$

For a desired trajectory which requires the quadrotor to follow a straight line path in the cartesian workspace, the system's simulated trajectory is shown in Fig. 4.

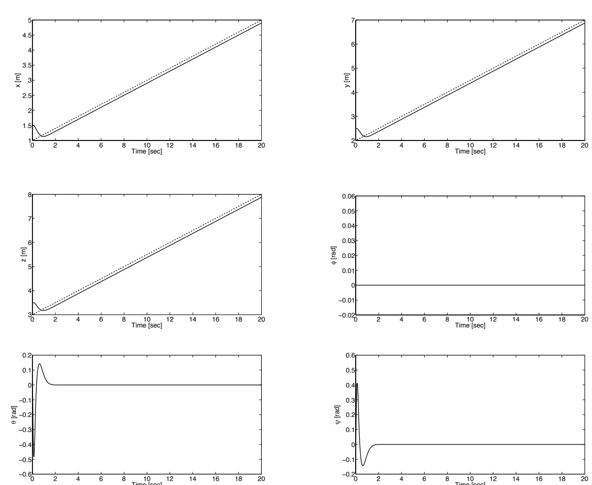

Fig. 4. Quadrotor position and attitude using the two-level hierarchical controller. The dash lines are the desired trajectories for flat outputs $y_d = \eta_d = z$.

B. Three level Hierarchical Controller

Taking into account the small angle errors, the additional control loop of the three-level hierarchical control architecture is now established. In this case, with $\lambda = 10$, the interface $u_v = Rv + Qz + K(\xi - Pz)$ is constructed with $R = 2.88 \times 10^3\, I_3$, $Q = \mathbf{0}_3$ and $K = \mathrm{diag}(K_b, K_b, K_b)$ with $K_b = 10^5 \times [-2.13, -0.41, -0.0281, -0.0007]$.

The pseudo-input u^F is constructed as

$$u_1^F = x_d^{(4)} - k_{1\,1} e_1^{(3)} - k_{1\,2} \ddot{e}_1 - k_{1\,3} \dot{e}_1 - k_{1\,4} e_1$$
$$u_2^F = y_d^{(4)} - k_{2\,1} e_1^{(3)} - k_{2\,2} \ddot{e}_2 - k_{2\,3} \dot{e}_2 - k_{2\,4} e_2$$
$$u_3^F = z_d^{(4)} - k_{3\,1} e_3^{(3)} - k_{3\,2} \ddot{e}_3 - k_{3\,3} \dot{e}_3 - k_{3\,4} e_3$$

where $e_1 = x - x_d$, $e_2 = y - y_d$ and $e_3 = z - z_d$. Note that $x_d^{(4)} = u_v(1)$, $y_d^{(4)} = u_v(2)$, $z_d^{(4)} = u_v(3)$. The desired values for the flat outputs and their derivatives are found using the state of flat reference system ξ_d. The gains of u^F are selected as $\begin{bmatrix} k_{i\,4} & k_{i\,3} & k_{i\,2} & k_{i\,1} \end{bmatrix} = \begin{bmatrix} 40 & 70 & 50 & 12 \end{bmatrix}$ for $i = 1, 2, 3$ and $\begin{bmatrix} k_{\phi 2} & k_{\phi 1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$. Fig. 5 shows the evolution of the state variables $[x, y, z, \phi, \theta, \psi]^T$ as they are steered from $x_e(t_0)$ to $x(t_f)$ with $t_0 = 0$s and $t_f = 20$s.

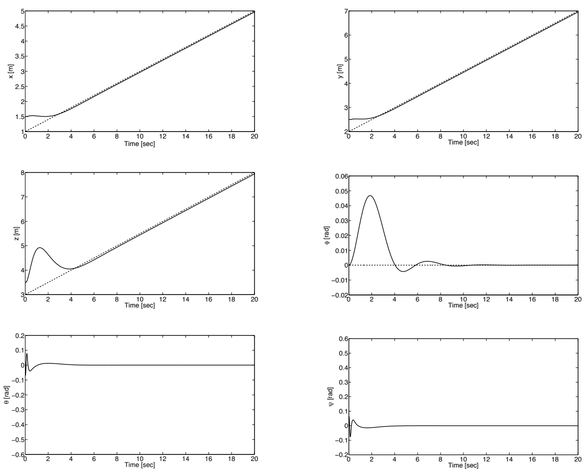

Fig. 5. Position and attitude of the quadrotor using the three-level hierarchical controller. The dash lines are the desired trajectories for flat outputs $y_d = \eta_d = z$.

We performed simulations to study the sensitivity of the controllers with respect to constant noise. In case of two-level controller, the system remains stable with noise of magnitute up to 0.7 in all states with $\lambda = 1.0$. In case of three-level controller, the system remains stable with noise of magnitute up to 0.4 in all states with $\lambda = 20$ (both controllers performed well under different λ). More thorough analysis of these methods under noise will be done in the future work.

For practical realization, one needs to measure the complete state vector for feedback. The flat outputs and their derivatives are computed using $\xi = T(x)$. Comparing the performance of the two control hierarchies, we find that the distance between two output trajectories of the abstract system and the original quadrotor system in Fig. 4 is larger than in Fig. 5. This is due to the usage of different values for the λ parameters. If λ is chosen to be the same, we would obtain the same bound on the distance between the outputs between the abstraction and the nonlinear one, as well as similar system performance. However, because of the simplification that were made in the dynamics of the nonlinear system, two-level hierarchical control design cannot afford $\lambda \geq 5$, while the three-level hierarchical control design can allow $\lambda > 20$ and achieve a smaller difference between the two output trajectories. This difference between our two control architectures requires further investigation.

V. CONCLUSION

Feedback linearizable (statically or dynamically) nonlinear systems admit lower dimensional linear abstractions. The paper offers a procedure for obtaining these abstractions, by exploiting the constructive methods available for linear systems. This procedure allows a trajectory to be planned on a low dimensional space, and then a control input to be constructed in order for a refinement of this trajectory to be implemented on the concrete nonlinear system. Depending on the nonlinear system's ability to be fully feedback linearized, two hierarchical control design architectures are presented. The first assumes that the nonlinear system is differentially flat, while the second addresses the case where reasonable modeling simplifications to the nonlinear dynamics can produce a differentially flat system. In the second case, a control loop is introduced to compensate for the approximation errors. The approaches are implemented on the dynamical model of a quadrotor and simulation results show that the trajectory of nonlinear system remains bounded with respect to abstract trajectory where the bound is given by the approximate simulation function between the feedback linearized system and the abstract system. A challenge left to be addressed is to determine how uncertain or stochastic nonlinear systems can be controlled through a similar type of hierarchical abstraction design.

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