GENERAL FORMULATION OF FLUID MECHANICS (Eulerian formulation)

LAMINAR FLOW OF NEWTONIAN FLUID: Eulerian description

Material derivative or substantial derivative:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f.$$

(1) Mass conservation:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

(2) Conservation of linear momentum:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \rho \mathbf{f} - \nabla p + \nabla \cdot [\tilde{\tau}]$$

where $\rho(\mathbf{x}, t)$: density, $\mathbf{v}(\mathbf{x}, t)$: velocity, $p(\mathbf{x}, t)$: pressure, $T(\mathbf{x}, t)$: temperature.

Total stress $\tau = \tilde{\tau} - pI$.

Constitutive Eqn

$$[\tilde{\tau}] = f([s])$$

where $[\tilde{\tau}]$: shear stress field,

$[s]$: rate-of-strain field, $s_{ij} = 0.5(\partial v_i / \partial x_j + \partial v_j / \partial x_i)$.

Examples: Newtonian fluid:

$$[\tilde{\tau}] = 2\mu[s] + \lambda \nabla \cdot \mathbf{v}[I]$$

$-$for Stokes fluid $\lambda = -2\mu/3$.

Viscoelastic fluid:

$$[\tilde{\tau}] = 2\mu[s] + [\tau^E]$$

where $[\tau^E]$: elastic contribution, nonlinear function of $[s]$ or $\nabla \mathbf{v}$.

(3) Energy conservation (1st law of thermodynamics)

$$\rho C_p \left[\frac{\partial T}{\partial t} + \underbrace{\mathbf{v} \cdot \nabla T}_{convection} \right] = \underbrace{k\nabla^2 T}_{conduction} + \underbrace{H}_{heat\ generation} - \underbrace{P\nabla \cdot \mathbf{v}}_{pressure\ work}$$

where heat generation $H = q_s + q_D + q_k + q_J + q_p$

q_s: due to applied heat sources or sinks,

$q_D = 2\mu[s] : [s]$: due to viscous dissipation (fluid elements rubbing against each other),

q_R: due to chemical reaction,

q_J: due to electrical heating,

q_p: thermal radiation.

(4) Species transport eqn. (e.g. for a reacting system, fluid=$N_2 + O_2 + H_2 + H_2O + ...$)

$$\rho \left(\frac{\partial c^i}{\partial t} + \mathbf{v} \cdot \nabla c^i \right) = \rho\alpha^i \nabla^2 c^i + source/sink + chemical\ reaction\ rate$$

where c^i: concentration of species i.

(5) Eqn of state:

$$f(P, \rho, T, c^i) = 0$$

examples: for ideal single-component gas

$$P = \rho RT = \rho \frac{R_0}{M} T$$

where R_0: universal gas constant, M: molecular weight.

For gas mixture, $1/M = \sum_i c^i/M^i$, where M^i: molecular weight of i-species.

(6) Boundary Conditions and Initial Conditions: $\mathbf{v}(s,t)$ or $[\tau](s,t)$: $s-$ boundary curve/surface

Normally the boundary condition for p is not required, except a reference value for it to be determined uniquely.

$c^i(s,t)$ or mass flux $= -\rho\alpha^i \mathbf{n} \cdot \nabla c^i$

$T(s,t)$ or heat flux $= -k\mathbf{n} \cdot \nabla T$

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Initial condition: $\mathbf{v}(\mathbf{x}, t=0)$, $c^i(\mathbf{x}, t=0)$, $T(\mathbf{x}, t=0)$, ...

(7) Problem formulation or physical modeling involves making assumptions to simplify the flow eqns.

Examples: isothermal, incompressible

How to effectively use these coupled PDEs to understand fluid physics and to aid engineering analysis?

Physical intuition and experimental observations

General theory of PDEs

Numerical solutions

TURBULENCE MODELING:

A state of flow that is highly random, unsteady, and three dimensional.

Too expensive to solv the local, time-dependent flow field (a range of scales in space and time)

What if only solving the time-averaged flow field

$$\eta \equiv \begin{pmatrix} \mathbf{v} \\ \rho \\ T \\ c^i \end{pmatrix}$$

mean/average field

time average:

$$\bar{\eta} \equiv \frac{1}{\triangle t} \int_t^{t+\triangle t} \eta dt$$

Reynolds decomposition:

$$\eta = \bar{\eta} + \hat{\eta}$$

where $\bar{\eta}$: averaged field (solved), $\hat{\eta}$: fluctuation field (modeled).

Take, for example, incompressible isothermal flow The exact eqns

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \mu \nabla^2 \mathbf{v}$$

$$\rho C_p \left[\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right] = k \nabla^2 T$$

For $\bar{u}, \bar{v}, \bar{w}, \bar{T}, \bar{P}$

$$\nabla \cdot \bar{\mathbf{v}} = 0 \rightarrow \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{w}}{\partial x}$$

x-component momentum eqn

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} \right) = -\frac{\partial \bar{P}}{\partial x} + \mu \nabla^2 \bar{u} - \frac{\partial(\rho\overline{\hat{u}\hat{u}})}{\partial x} - \frac{\partial(\rho\overline{\hat{u}\hat{v}})}{\partial y} - \frac{\partial(\rho\overline{\hat{u}\hat{w}})}{\partial z}$$

$$\rho C_p \left(\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z} \right) = k\nabla^2 \bar{T} - \frac{\partial(\rho C_p \overline{\hat{u}\hat{T}})}{\partial x} - \frac{\partial(\rho C_p \overline{\hat{v}\hat{T}})}{\partial y} - \frac{\partial(\rho C_p \overline{\hat{w}\hat{T}})}{\partial x}$$

Need modeling, namely, relating

$-\rho\overline{\hat{u}\hat{u}}$	$-\overline{\hat{u}\hat{T}}$	turbulence	$\overline{u}$
$-\rho\overline{\hat{u}\hat{v}}$	$-\overline{\hat{v}\hat{T}}$	$\Longleftrightarrow$	$\overline{v}$
$-\rho\overline{\hat{u}\hat{w}}$	$-\overline{\hat{w}\hat{T}}$	modeling	$\overline{w}$
			$\bar{T}$
Reynolds stress	thermal turbulent flux		

Examples

$$-\rho\overline{\hat{u}\hat{v}} = \mu_t \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)$$

where μ_t: turbulent viscosity.

$$-\overline{\hat{u}\hat{T}} = \alpha_t \left(\frac{\partial \bar{T}}{\partial x} \right)$$

where α_t: turbulent diffusivity.

<u>Turbulence models included in FIDAP</u>

Zero-equation model

$$\mu_t = \rho l_m^2 \left[[\bar{s}] : [\bar{s}] \right]^{1/2}$$

where l_m: mixing length, $[\bar{s}]$: averaged rate-of-strain field.

Two-eqn or $K - \epsilon$ model

Turbulent kinetic energy $k \equiv \frac{1}{2}(\overline{\hat{u}^2} + \overline{\hat{v}^2} + \overline{\hat{w}^2}$.

viscous dissipation of turbulent kinetic energy $\epsilon \equiv 2\nu[\hat{s}] : [\hat{s}]$.

Solve modeled transport eqns for K, ϵ

$$\mu_t = \rho c_\mu k^2 / \epsilon$$

where empirical constant $c_\mu \approx 0.09$

(more details can be found in FIDAP theory mannuals and "turbulence" textbook.)

For a layman, the mean fields in turbulent flow are subject to spatially varying variable effective viscosity!