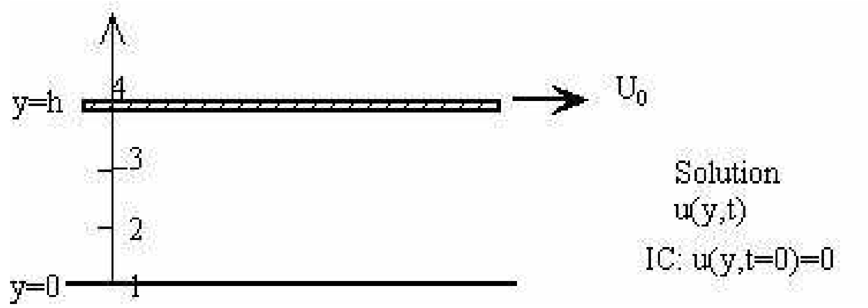
FINITE-ELEMENT FORMULATION OF AN UNSTEADY 1D FLOW PROBLEM



The problem:

Fluid motion in between two plates due to sudden movement of top plate at $t = 0$.

Assumptions:

Uniform density, pressure, Newtonian fluid, isothermal

Governing Eqn: The Navier-Stokes Eqns can be reduced to only one equation for $u(y,t)$

$$\begin{array}{ccc} \rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} & \theta \equiv u - \frac{y}{h} U_0 & \rho \frac{\partial \theta}{\partial t} = \mu \frac{\partial^2 \theta}{\partial y^2} \\ u(y, t=0) = 0 & \Longrightarrow & \theta(y, t=0) = -yU_0/h \\ u(y=0, t) = 0 & & \theta(y=0, t) = 0 \\ u(y=h, t) = U_0 & & \theta(y=h, t) = 0 \end{array}$$

Transformation of $u(y,t)$ to $\theta(y,t)$ simplifies the BC's.

Separation of variables for θ

$$\theta = \sum_{k=1}^{\infty} A_k \exp(-\frac{\mu}{\rho} \frac{\pi^2 k^2}{h^2} t) \sin(\frac{\pi k y}{h})$$

where

$$A_k = \frac{2}{h} \int_0^h -\frac{y}{h} U_0 \sin(\frac{\pi k y}{h}) dy = \frac{2U_0}{\pi k} (-1)^k$$

$$u(y,t) = \frac{y}{h} U_0 + \sum_{k=1}^{\infty} (-1)^k \frac{2U_0}{\pi k} \sin(\frac{\pi k y}{h}) \exp(-\frac{\mu}{\rho} \frac{\pi^2 k^2}{h^2} t)$$

For large t or $t \gg \rho h^2/\mu$ (diffusion time)

$$u(y,t) \approx \frac{y}{h} U_0 - \frac{2U_0}{\pi} \sin(\frac{\pi y}{h}) \exp(-\frac{\mu}{\rho} \frac{\pi^2}{h^2} t)$$

or in non-dimensional form:

$$\frac{u}{U_0} = f\left(\frac{y}{h}, \frac{\mu t}{\rho h^2}\right)$$

Finite-element solution with 3 equal elements $l = h/3$. Solution in terms of θ:

$$\theta \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} (t)$$

$$\int_0^h dy\psi(y)\left[\rho\frac{\partial\theta}{\partial t} - \mu\frac{\partial^2\theta}{\partial y^2}\right] = 0$$

$$\underbrace{\frac{\partial}{\partial t}\int_0^h \rho\theta\psi dy}_{(I)} - \underbrace{\psi(y)\mu\frac{\partial\theta}{\partial y}|_0^h}_{(II)} + \underbrace{\int_0^h \mu\frac{\partial\theta}{\partial y}\frac{\partial\psi}{\partial y}dy}_{(III)} = 0$$

where $(II) = 0$ due to BC's. (note that BC's for ψ are the same as θ)

$$(I) = \frac{\partial}{\partial t}\sum_{(e)}\int_0^l dy_1\rho[\psi_1(1-\frac{y_1}{l}) + \psi_2\frac{y_1}{l}][\theta_1(1-\frac{y_1}{l}) + \theta_2\frac{y_1}{l}]$$

$$(I) = \frac{\partial}{\partial t}(\psi_1,\psi_2,\psi_3,\psi_4)\frac{\rho l}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$$(III) = \sum_{(e)}\int_0^l \mu\frac{\psi_2-\psi_1}{l}\frac{\theta_2-\theta_1}{l}dy$$

$$(III) = \frac{\mu}{l}(\psi_1,\psi_2,\psi_3,\psi_4)\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}$$

$\Longrightarrow$

$$\Psi^T\left[M\frac{dQ}{dt} + kQ\right] = 0$$

But $\psi_1 = \psi_4 = 0$, so only the 2nd and 3rd eqns in $M\frac{dQ}{dt} + kQ = 0$ are needed.

$$\frac{\rho l}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} \frac{d\theta_2}{dt} \\ \frac{d\theta_3}{dt} \end{bmatrix} + \frac{\mu}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

$\Longrightarrow$

$$\begin{bmatrix} \frac{d\theta_2}{dt} \\ \frac{d\theta_3}{dt} \end{bmatrix} = \frac{54\mu}{5\rho h^2}\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix}$$

Define $\alpha \equiv \frac{54\mu}{5\rho h^2}$ Solution

$$\begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}$$

here λ is eigenvalue, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is eigenvector.
$\Longrightarrow$

$$\lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \alpha \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{vmatrix} -3 - \frac{\lambda}{\alpha} & 2 \\ 2 & -3 - \frac{\lambda}{\alpha} \end{vmatrix} = 0$$

$\Longrightarrow \lambda_1 = -5\alpha,\ \lambda_2 = -2$
eigenvector for λ_1 is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

eigenvector for $\lambda_2 1$ is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution is

$$\begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5\alpha t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\alpha t}$$

$t = 0$: $-\frac{1}{3}U_0 = c_1 + c_2$
$-\frac{2}{3}U_0 = -c_1 + c_2$
$\Longrightarrow$
$c_1 = U_0/6$, $c_2 = -U_0/2$. Namely

$$\frac{u_2}{U_0} = \frac{1}{3} + \frac{1}{6}e^{-5\alpha t} - \frac{1}{2}e^{-\alpha t}$$

$$\frac{u_3}{U_0} = \frac{2}{3} - \frac{1}{6}e^{-5\alpha t} - \frac{1}{2}e^{-\alpha t}$$

Exact solution u/U_0

y/h	$\frac{t}{h^2/\nu} = 0.01$	$\frac{t}{h^2/\nu} = 0.1$	$\frac{t}{h^2/\nu} = 1.0$
1/3	2.4603e-6	0.1332	0.3333
2/3	1.8422e-2	0.4559	0.6666

Finite element with 3 linear elements

y/h	$\frac{t}{h^2/\nu} = 0.01$	$\frac{t}{h^2/\nu} = 0.1$	$\frac{t}{h^2/\nu} = 1.0$
1/3	3.6240e-6	0.2572	0.3333
2/3	0.1285	0.5775	0.6667

Note that in the above, the time integration was done exactly for the finite element formulation. In a software package like FIDAP, this is done by numerical approximation.

TIME INTEGRATION SCHEMES

Finite element formulation of unsteady/dynamic problems will result in Partial DE – a set of coupled Ordinary differential eqns.

$$A\frac{du}{dt} + Ku = f$$

where A and K are matrices

$$u = \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ u_n \end{pmatrix}$$

is solution vector.

$$f = \begin{pmatrix} f_1 \\ f_2 \\ . \\ . \\ f_n \end{pmatrix}$$

is forcing vector, with IC for f, one needs to integrate in time.

Consider a one-degree-of-freedom ODE

$$\frac{du}{dt} = f(u(t), t) \equiv F(t)$$

Let the exact soln be $u(t)$, which we usually do not know.

Numerical approximation for $u|_{t=t_k} \approx u(t_k) \to u^k$

Finite difference formulation:
Assume $u^0, u^1, ..., u^k$ are known numerically, how to obtain u^{k+1}?

$$\frac{u^{k+1} - u^k}{\triangle t} = aF^{k+1} + bF^k + cF^{k-1} + ...$$

If $a \neq 0$, implicit scheme;
$a = 0$, explicit scheme.

How to determine a, b, c ?

Taylor expansion

$$u^{k+1} \approx u(t^{k+1}) = u(t^k + \triangle t) = u^k + \frac{du}{dt} \triangle t + \frac{d^2u}{dt^2}\frac{\triangle t^2}{2} + \triangle(\triangle t^3)$$

$$\frac{u^{k+1} - u^k}{\triangle t} = \frac{du}{dt} + \frac{d^2u}{dt^2}\frac{\triangle t}{2} + \frac{d^3u}{dt^3}\frac{\triangle t^2}{6} + \triangle(\triangle t^3)$$

$$\frac{u^{k+1}-u^k}{\triangle t} = F + F'\frac{\triangle t}{2} + F''\frac{\triangle t^2}{6} + \triangle(\triangle t^3)$$

$$aF^{k+1} + bF^k + cF^{k-1} = a(F^k + F'\triangle t + F''\frac{\triangle t^2}{2} + \triangle(\triangle t^3)) + bF^k + c(F^k - F'\triangle t + F''\frac{\triangle t^2}{2} + \triangle(\triangle t^3))$$

Then the leading order $1 = a + b + c$,
$O(\triangle t)$: $\frac{1}{2} = a - c$,
$O(\triangle t^2)$: $\frac{1}{6} = \frac{a}{2} + \frac{c}{2}$ and so on.

The 1st-order scheme:
Only the leading order is balanced.
Ex. $a = c = 0,\ b = 1$
$\frac{u^{k+1}-u^k}{\triangle t} = F^k$: forward Euler scheme
Ex. $a = 1,\ b = c = 0$
$\frac{u^{k+1}-u^k}{\triangle t} = F^{k+1}$: backward Euler scheme (implicit)

The 2nd-order scheme: coefficients up to $O(\triangle t)$ are balanced.
Ex. $a = 0,\ b = 3/2,\ c = -1/2$
$\frac{u^{k+1}-u^k}{\triangle t} = \frac{3}{2}F^k - \frac{1}{2}F^{k-1}$: 2nd-order Adams-Bashforth scheme.
Ex. $a = 1/2,\ b = 1/2,\ c = 0$
$\frac{u^{k+1}-u^k}{\triangle t} = \frac{1}{2}F^{k+1} + \frac{1}{2}F^k$: 2nd-order Crank-Nicholson scheme or Trapezoidal scheme.

Using the above procedure, one can develop high-order scheme.

FIDAP only implemented
Forward Euler, 1st/explicit
Backward Euler, 1st/implicit
Trapezoidal scheme, 2nd/implicit

ACCURACY/ TRUNCATION ERROR

Consider forward Euler scheme: $\frac{u^{k+1}-u^k}{\triangle t} = F^k = f(u^k, t^k)$

$$u^{k+1} = u^k + \triangle t f(u^k, t^k)$$

But

$$u(t^{k+1}) = u(t^k) + \frac{du}{dt}|_{t=t^k} \triangle t + O(\triangle t^2)$$

$$u(t^{k+1}) = u(t^k) + \triangle t f(u(t^k), t^k) + O(\triangle t^2)$$

Then

$$u(t^{k+1}) - u^{k+1} = u(t^k) - u^k + \triangle t[f(u(t^k), t^k) - f(u^k, t^k)] + O(\triangle t^2)$$

Define truncation error:

$$\epsilon^k = u(t^k) - u^k = exact - numericalapprox.$$

Assume $f(u,t)$ is a well-behaved function ($\partial f/\partial u$ is finite)

$$|f(u(t^k),t^k) - f(u^k,t^k)| \leq A|u(t^k) - u^k| = A|\epsilon^k|$$

$$\begin{aligned}
|\epsilon^{k+1}| &\leq |\epsilon^k| + \triangle t A|\epsilon^k| + R \\
&= (1 + A\triangle t)|\epsilon^k| + R \\
&= (1 + A\triangle t)^2|\epsilon^{k-1}| + (1 + A\triangle t)R + R \\
&\leq (1 + A\triangle t)^{k+1}|\epsilon^0| + R\sum_{i=0}^{k}(1 + A\triangle t)^i \\
&= k\frac{(1 + A\triangle t)^{k+1} - 1}{A\triangle t}
\end{aligned}$$

Where $|\epsilon^0|$ is assumed zero initially.

$$|\epsilon^N| \leq R\frac{1}{A\triangle t}\left[\left(1 + \frac{AT}{N}\right)^N - 1\right] \leq \frac{R}{A\triangle t}(e^{LT} - 1)$$

Since $e^{LT} - 1$ is fixed, does not change with $\triangle t$

$$|\epsilon^N| = O(\triangle t)$$

This is why Euler scheme is known as 1st order method.

Convergence is obtained as $\triangle t \to 0$.

Stability:

A stable numerical scheme is one for which errors from any source (round-off, truncation, mistakes) do not grow in the sequence of numerical procedures as the calculation proceeds from one time step to the next.

Ex. Consider the differential eqn

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

where c is some characteristic velocity.

Let u_j^n represent numerical approximation to $u(x_j, t^n)$. Use forward Euler scheme in time

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\frac{\partial u}{\partial x}|^n = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

We have

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

Question: Is this numerical scheme stable?

Let numerical soln to the difference eqn be N,

exact soln to the difference eqn be D,
round-off error be ϵ

$$N = D + \epsilon$$

Then ϵ satisfies the same eqn as u.

$$\epsilon_j^{n+1} = \epsilon_j^n - \frac{\alpha}{2}(\epsilon_{j+1}^n - \epsilon_{j-1}^n)$$

where $\alpha \equiv c\Delta t/\Delta x$

Now assume that the error is of the form (stability analysis, Fourier analysis in space)

$$\epsilon_j^n = e^{at} e^{ikx_j}$$

$$e^{a(t^n+\Delta t)} e^{ikx_j} = a^{at^n} e^{ikx_j} - \frac{\alpha}{2}(e^{ik\Delta x} - e^{-ik\Delta x}) e^{at^n} e^{ikx_j}$$

Growth factor

$$e^{a\Delta t} = 1 - \frac{\alpha}{2} 2i \cdot \sin(k\Delta x) = 1 - i\alpha \sin(k\Delta x)$$

$|1 - i\alpha \sin(k\Delta x)| < 1$ to be stable,
$\sqrt{1 + \alpha^2 \sin^2(k\Delta x)} < 1$ impossible for any α.

The scheme is unconditionally unstable.

Consider a modification to the scheme

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{c\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

Then the growth factor is $|\cos(k\Delta x) - i\alpha \sin(k\Delta x)| < 1$
$\sqrt{\cos^2(k\Delta x) + \alpha^2 \sin^2(k\Delta x)} < 1$
$\sqrt{\frac{1+\cos(2k\Delta x)}{2} + \frac{\alpha^2}{2}(1 - \cos(2k\Delta x)} < 1$
$\sqrt{\frac{1+\alpha^2}{2} + \frac{1-\alpha^2}{2} \cos(2k\Delta x)} < 1$
Satisfied if $\alpha < 1$.

The new scheme is conditionally stable.

$$\alpha = c\Delta t/\Delta x < 1$$

known as CFL condition, Courant-Friedrichs-Lewy Condition.

How about using implicit (backward) Euler scheme?

$$u_j^{n+1} = u_j^n - \frac{\alpha^2}{2}(u_{j+1}^{n+1} - u_{j-1}^{n+1})$$

$$e^{a\Delta t} = 1 - i\alpha \sin(k\Delta x) e^{a\Delta t}$$

$$|e^{a\Delta t}| = \left|\frac{1}{1 + i\alpha\sin(k\Delta x)}\right| \leq 1$$

$$\frac{1}{\sqrt{1 + \alpha^2 \sin^2(k\Delta x)}} \leq 1$$

Always true. Therefore the backward Euler scheme is unconditionally stable.

Considerations for the choice of Δt: compromise the following factors

numerical truncation error (small Δt)
numerical stability (small Δt)
round-off error (large Δt)
total run time (perfer large Δt)