Fundamental Concepts in Continuum Mechanics
(As applied to mutli-dimensional solids and fluids)

A. Lagrangian vs Eulerian descriptions

Consider a "particle" (or tiny material element) within a 3D body. Let $\vec{X}$ be the position of this particle at $t=0$. The position of this particle at a later time t is $\vec{x}$

$$\vec{x}=\vec{x}\left(\vec{X},t\right) \qquad \text{given that} \quad \vec{x}\left(\vec{X},t=0\right)=\vec{X}$$

Consider a physical variable q, say temperature, that is associated with this particle.

$q=q(\vec{X},t)$ temperature at t of a given material particle initially located at $\vec{X}$

This is material description or Lagrangian description.

We may also choose to observe the changes at fixed spatial location $\vec{x}$

$$q=q(\vec{x},t)$$

This is spatial or Eulerian description.

NOTE: A given spatial location $\vec{x}$ is occupied by different particles at different times.

B. Deformation and displacement

Displacement of a particle $\vec{u}(\vec{X},t)\equiv\vec{x}(\vec{X},t)-\vec{X}$

The Lagrangian strain tensor $e_{ij}=\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial X_j}+\dfrac{\partial u_j}{\partial X_i}\right)$, symmetric

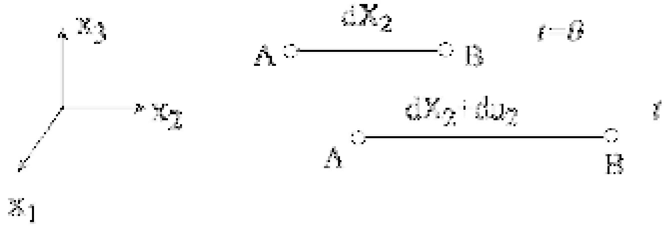
$$[e]=\begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2}+\dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3}+\dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2}+\dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3}+\dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3}+\dfrac{\partial u_3}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3}+\dfrac{\partial u_3}{\partial X_2}\right) & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix}$$

Notation: $\vec{u}=(u_1,u_2,u_3)=u_i=(u,v,w)$, spatial location: $(x_1,x_2,x_3)=(x,y,z)$

Geometrical interpretation of $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$

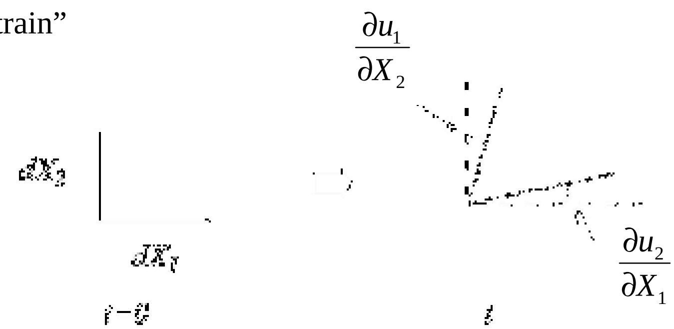
$$e_{22} = \frac{\partial u_2}{\partial X_2} = \frac{\text{Change of length for a material element aligned in the } x_2 \text{ direction}}{\text{Original length of the same material element}}$$

Also known as “normal strain” or “unit elongation” in the x_2 direction.



$$2e_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = \text{decrease in angle between two elements initially aligned in the } x_1 \text{ and } x_2 \text{ direction}$$

Known as “shear strain”



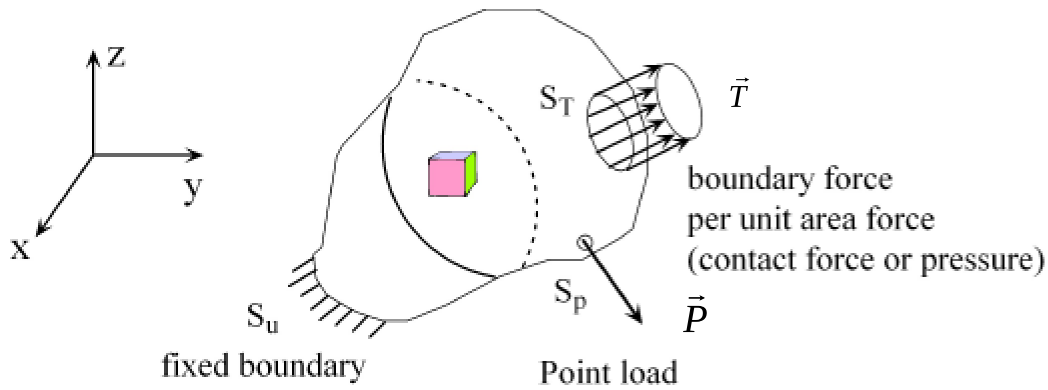
The trace of e_{ij}

$$= e_{11} + e_{22} + e_{33} = \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} = \frac{d(dV)}{dV} = \frac{\text{change of material volume}}{\text{original material volume}}$$

Known as “dilatation”.

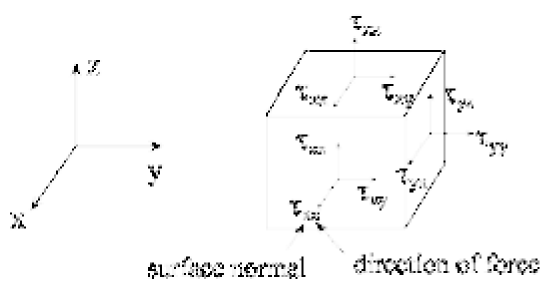
C. Stresses and equilibrium

Consider a 3D body occupying a volume *V* and having a surface *S*



distributed force per unit volume (weight) $\vec{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$

The stresses acting on an elemental volume (force per unit area)



The principle of moment of momentum $\rightarrow$ symmetry of stress tenor

The moment of all the forces acting on the elemental volume
= moment of inertia $\times$ angular acceleration

$$t_{zx} = t_{xz}$$
$$t_{yx} = t_{xy}$$
$$t_{yz} = t_{zy}$$

The statics or equilibrium equations

$\sum$ all forces acting on an element volume $= \Delta m \times$ acceleration

$$\sum F_x = 0: \qquad \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} + f_x = 0$$

$$\sum F_y = 0: \qquad \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z} + f_y = 0$$

$$\sum F_z = 0: \qquad \frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} + f_z = 0$$

Boundary conditions:

$$\vec{u} = 0 \quad \text{on} \quad S_u$$

$$\left.\begin{aligned} t_{xx}n_x + t_{xy}n_y + t_{xz}n_z &= T_x \\ t_{xy}n_x + t_{yy}n_y + t_{yz}n_z &= T_y \\ t_{xz}n_x + t_{yz}n_y + t_{zz}n_z &= T_z \end{aligned}\right\} \quad \text{on} \quad S_T \qquad \text{surface normal} \quad \vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

Similar for localized load or point load S_P, except over small area.

D. Stress-strain relations for linear elastic solid: The generalized Hooke's Law

For isotropic materials, only two materials properties are needed:
Young's modulus (or modulus of elasticity) E
Poisson's ratio n

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{t_{xx}}{E} - n\frac{t_{yy}}{E} - n\frac{t_{zz}}{E}$$

$$e_{yy} = \frac{\partial \mathrm{v}}{\partial y} = -n\frac{t_{xx}}{E} + \frac{t_{yy}}{E} - n\frac{t_{zz}}{E}$$

$$e_{zz} = \frac{\partial w}{\partial z} = -n\frac{t_{xx}}{E} - n\frac{t_{yy}}{E} + \frac{t_{zz}}{E}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right) = \frac{t_{xy}(1+n)}{E}$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial \mathrm{v}}{\partial z} + \frac{\partial \mathrm{w}}{\partial y}\right) = \frac{t_{yz}(1+n)}{E}$$

$$e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial \mathrm{w}}{\partial x}\right) = \frac{t_{xz}(1+n)}{E}$$

Inverse relations

$$t_{xx} = \frac{E}{(1+n)(1-2n)}\left[(1-n)\frac{\partial u}{\partial x} + n\frac{\partial \mathrm{v}}{\partial y} + n\frac{\partial w}{\partial z}\right]$$

$$t_{yy} = \frac{E}{(1+n)(1-2n)}\left[n\frac{\partial u}{\partial x} + (1-n)\frac{\partial \mathrm{v}}{\partial y} + n\frac{\partial w}{\partial z}\right]$$

$$t_{zz} = \frac{E}{(1+n)(1-2n)}\left[n\frac{\partial u}{\partial x} + n\frac{\partial \mathrm{v}}{\partial y} + (1-n)\frac{\partial w}{\partial z}\right]$$

$$t_{xy} = \frac{E}{2(1+n)}\left[\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right]$$

$$t_{yz} = \frac{E}{2(1+n)}\left[\frac{\partial w}{\partial y} + \frac{\partial \mathrm{v}}{\partial z}\right]$$

$$t_{xz} = \frac{E}{2(1+n)}\left[\frac{\partial u}{\partial z} + \frac{\partial \mathrm{w}}{\partial x}\right]$$

Von Mises stress $s_{VM} \equiv \sqrt{I_1^2 - 3I_2}$

where $I_1 = t_{ii} = t_{xx} + t_{yy} + t_{zz}$, $I_2 = \frac{1}{2}I_1^2 - t_{ij}t_{ji}$

often used as a criterion in determining the onset of failure in materials

E. Special cases:

One dimensional models:

(a) If $u = u(x)$, $t_{xx} = \frac{E}{(1+n)(1-2n)}\left[(1-n)\frac{du}{dx}\right] \approx E\frac{du}{dx}$.

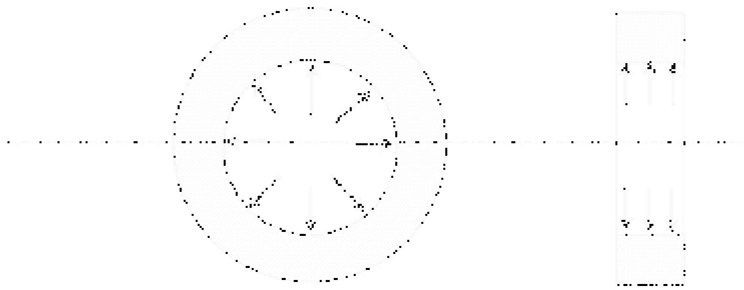
(b) If $\mathrm{v} = \mathrm{v}(x)$ and $u = w = 0$, then $t_{xy} = \frac{E}{2(1+n)}\frac{d\mathrm{v}}{dx}$.

Two dimensional models:

(a) Plane stress model: Only in-plane stresses are nonzero. $t_{zz} = 0$, $t_{xz} = 0$, $t_{yz} = 0$

Good for thin planar body subjected to in-plane loading.

Example: A thin ring press fitted on a shaft.



$$e_{xx} = \frac{t_{xx}}{E} - n\frac{t_{yy}}{E}$$

$$e_{yy} = -n\frac{t_{xx}}{E} + \frac{t_{yy}}{E}$$

$$e_{xy} = \frac{t_{xy}(1+n)}{E}$$

$$e_{zz} = -\frac{n}{E}\left(t_{xx} + t_{yy}\right)$$ non-zero off-plane strain

(b) Plane-strain model: Only in-plane strains are nonzero. $e_{zz} = 0, \quad e_{xz} = 0, \quad e_{yz} = 0$

But $t_{zz} \neq 0$

Good for a long body of uniform cross section subjected to uniform transverse loading along its length.