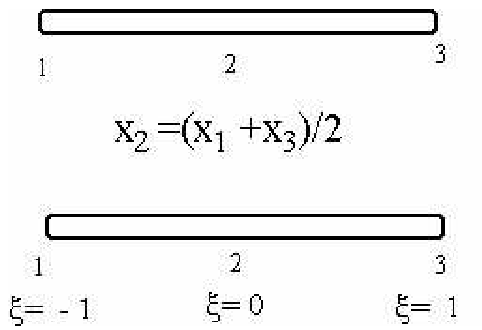
ADVANCED FEATURES OF 1D FE MODELING

−Illustrate the use of high-order elements

USE OF QUADRATIC ELEMENTS

A three-node quadratic element



Local coordinate:

$$\xi = \frac{2(x - x_2)}{(x_3 - x_1)}$$

Shape functions:

$$\begin{aligned}
N_1(\xi) &= -\frac{1}{2}\xi(1-\xi) \\
N_2(\xi) &= (1+\xi)(1-\xi) \\
N_3(\xi) &= \frac{1}{2}\xi(1+\xi)
\end{aligned}$$

	$\xi = -1$	$\xi = 0$	$\xi = 1$
$N_1(\xi)$	1	0	0
$N_2(\xi)$	0	1	0
$N_3(\xi)$	0	0	1

The displacement in the element is interpolated as

$$\begin{aligned}
q(x) &= N_1 q_1 + N_2 q_2 + N_3 q_3 \\
&= (N_1,\, N_2,\, N_3) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}
\end{aligned}$$

Strain

$$\begin{aligned}
\epsilon &= \frac{dq}{dx} = \frac{dq}{d\xi} \Big/ \frac{dx}{d\xi} = \frac{2}{x_3 - x_1} \left(\frac{dN_1}{d\xi},\, \frac{dN_2}{d\xi},\, \frac{dN_3}{d\xi} \right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \\
&= \frac{2}{l^{(e)}} \left(-\frac{1-2\xi}{2},\, -2\xi,\, \frac{1+2\xi}{2} \right) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \\
&= \mathbf{B} \cdot \mathbf{q}
\end{aligned}$$

and the stress $\tau = E\mathbf{B}\cdot\mathbf{q}$.

Now stress and strain can vary linearly within an element.

USE OF TWO QUADRATIC ELEMENTS FOR THE STEEL PLATE PROBLEM

$$-\int_0^{24} AE\frac{dq}{dx}\frac{d\phi}{dx}dx + \int_0^{24}\phi\rho A(x)dx + \phi(x=12)P = 0$$

Then

$$\begin{aligned}
\int_0^{24} AE\frac{dq}{dx}\frac{d\phi}{dx}dx &= \frac{5.25E}{l^{(e)}}\cdot 2\cdot\int_{-1}^{1}\left[\frac{2\xi-1}{2}q_1 - 2\xi q_2 + \frac{1+2\xi}{2}q_3\right]\left[\frac{2\xi-1}{2}\phi_1 - 2\xi\phi_2 + \frac{1+2\xi}{2}\phi_3\right]d\xi \\
&\quad + \frac{3.75E}{l^{(e)}}\cdot 2\cdot\int_{-1}^{1}\left[\frac{2\xi-1}{2}q_3 - 2\xi q_4 + \frac{1+2\xi}{2}q_5\right]\left[\frac{2\xi-1}{2}\phi_3 - 2\xi\phi_4 + \frac{1+2\xi}{2}\phi_5\right]d\xi \\
&= \frac{5.25E}{l^{(e)}}\cdot 2(\phi_1,\,\phi_2,\,\phi_3)\begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ \frac{1}{6} & -\frac{4}{3} & \frac{7}{6}\end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3\end{bmatrix} \\
&\quad + \frac{3.75E}{l^{(e)}}\cdot 2(\phi_3,\,\phi_4,\,\phi_5)\begin{bmatrix} \frac{7}{6} & -\frac{4}{3} & \frac{1}{6} \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ \frac{1}{6} & -\frac{4}{3} & \frac{7}{6}\end{bmatrix}\begin{bmatrix} q_3 \\ q_4 \\ q_5\end{bmatrix} \\
&= \frac{E}{3l^{(e)}}(\phi_2,\,\phi_3,\,\phi_4,\,\phi_5)\begin{bmatrix} 84 & -42 & 0 & 0 \\ -42 & 63 & -30 & 3.75 \\ 0 & -30 & 60 & -30 \\ 0 & 3.75 & -30 & 26.25\end{bmatrix}\begin{bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\int_0^{24}\phi\rho A(x)dx &= \frac{l_e}{2}\cdot 5.25\rho\int_{-1}^{1}(N_1\phi_1 + N_2\phi_2 + N_3\phi_3)d\xi \\
&\quad + \frac{l_e}{2}\cdot 3.75\rho\int_{-1}^{1}(N_1\phi_3 + N_2\phi_4 + N_3\phi_5)d\xi \\
&= \frac{l_e}{2}\cdot 5.25\rho\left[\frac{1}{3}\phi_1 + \frac{4}{3}\phi_2 + \frac{1}{3}\phi_3\right] \\
&\quad + \frac{l_e}{2}\cdot 3.75\rho\left[\frac{1}{3}\phi_3 + \frac{4}{3}\phi_4 + \frac{1}{3}\phi_5\right] \\
&= \frac{l_e\rho}{6}(\phi_2,\,\phi_3,\,\phi_4,\,\phi_5)\begin{pmatrix} 21 \\ 9 \\ 15 \\ 3.75\end{pmatrix}
\end{aligned}$$

$$\Longrightarrow$$

$$-\frac{E}{36}\begin{bmatrix} 84 & -42 & 0 & 0 \\ -42 & 63 & -30 & 3.75 \\ 0 & -30 & 60 & -30 \\ 0 & 3.75 & -30 & 26.25\end{bmatrix}\begin{bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5\end{bmatrix} + 2\rho\begin{bmatrix} 21 \\ 9 \\ 15 \\ 3.75\end{bmatrix} + \begin{bmatrix} 0 \\ P \\ 0 \\ 0\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0\end{bmatrix}$$

where $E = 30E+6$ psi, $\rho = 0.2836$ lbf/in^3, $P = 100$ lbf.

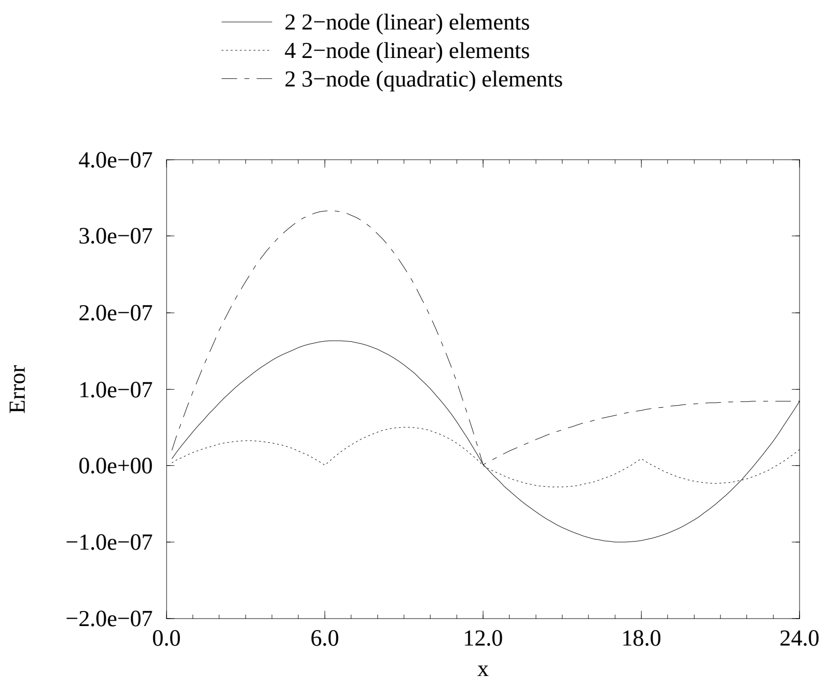
$$\frac{E}{36}\begin{bmatrix} 84 & -42 & 0 & 0 \\ -42 & 63 & -30 & 3.75 \\ 0 & -30 & 60 & -30 \\ 0 & 3.75 & -30 & 26.25\end{bmatrix}\begin{bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5\end{bmatrix} = \begin{bmatrix} 42\rho \\ 18\rho + P \\ 30\rho \\ 7.5\rho\end{bmatrix}$$

Put into a direct solver:

$$\begin{bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} 4.806175 \\ 9.272029 \\ 9.782509 \\ 9.952670 \end{bmatrix} e^{-6}$$

	$u(x=12) \times 10^6$	$u(x=24) \times 10^6$
Exact	9.2707	9.8684
linear elements ($N=2$)	9.27203	9.95267
linear elements ($N=4$)	9.27071	9.88948
quadratic elements ($N=2$)	9.27203	9.95267

A plot of solution error (q(x)-u(x)):



Note that the solution is less accurate if quadratic elements are used. This could be due to the fact that the exact solution is mostly linear than quadratic.

However, if we consider a slightly different problem, the quadratic elements can be more accurate.

Assume no external load $P = 0$, and the plate has a constant width of 4.5 in. Study the deformation under the body force.

Analytical solution:

$$4.5E\frac{du}{dx} = 4.5x(24 - x)\rho$$

$$u(x) = \frac{\rho}{E}(24x - \frac{x^2}{2})$$

Thus

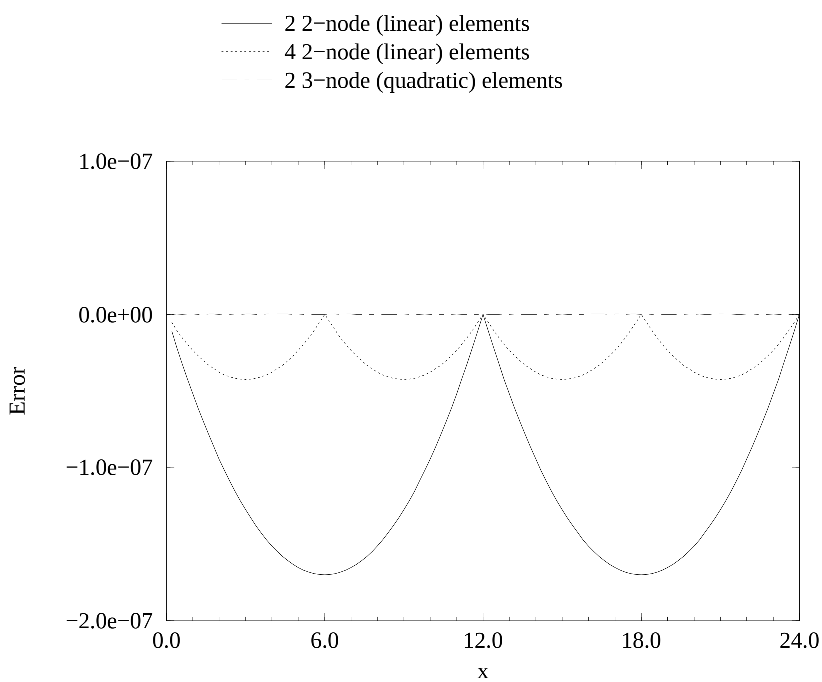
$$\begin{array}{ll} u(x=6) = 126\frac{\rho}{E}, & u(x=12) = 216\frac{\rho}{E}, \\ u(x=18) = 270\frac{\rho}{E}, & u(x=24) = 288\frac{\rho}{E}. \end{array}$$

Use two linear elements:

$$\begin{array}{lll} 4.5E\frac{q_1}{12} = 4.5\rho \times 18 & \rightarrow & q_1 = u(x=12) = 216\frac{\rho}{E}, \\ 4.5E\frac{q_2-q_1}{12} = 4.5\rho \times 6 & \rightarrow & q_2 = u(x=24) = 288\frac{\rho}{E}. \end{array}$$

Use 4 linear elements:

$$\begin{array}{lll} 4.5E\frac{q_1}{6} = 4.5\rho \times 21 & \rightarrow & q_1 = u(x=6) = 126\frac{\rho}{E} \\ 4.5E\frac{q_2-q_1}{6} = 4.5\rho \times 15 & \rightarrow & q_2 = u(x=12) = 216\frac{\rho}{E} \\ 4.5E\frac{q_3-q_2}{6} = 4.5\rho \times 9 & \rightarrow & q_3 = u(x=18) = 270\frac{\rho}{E} \\ 4.5E\frac{q_4-q_3}{6} = 4.5\rho \times 3 & \rightarrow & q_4 = u(x=24) = 288\frac{\rho}{E} \end{array}$$



Use 2 quadratic elements: after going through the similar procedure, we obtain:

$$\frac{E}{36}\begin{bmatrix} 72 & -36 & 0 & 0 \\ -36 & 63 & -36 & 4.5 \\ 0 & -36 & 72 & -36 \\ 0 & 4.5 & -36 & 31.5 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = 2\rho\begin{bmatrix} 18 \\ 9 \\ 18 \\ 4.5 \end{bmatrix}$$

Put into a direct solver:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 126 \\ 216 \\ 270 \\ 288 \end{bmatrix} \frac{\rho}{E}$$

All the methods give exact prediction at node points. The quadratic elements give exact solution at all x.

Conclusion: high-order elements usually give more accurate solution, but NOT always!

In general, the error $e(x)$ in a finite element approximation behaves like

$$|e(x)| \leq C \delta l^{n+1},$$

where δl is the distance between nodes, n is the oder of interpolation polynomial ($n = 1$ for linear elements, $n = 2$ for quadratic elements), C is a finite constant.