FINITE ELEMENT MODELING OF 1D PROBLEM
(Note the basic procedure is the same for 2- and 3-dimensional problems)

Lecture Objectives:

–Demonstrate 1D FE modeling using the stress analysis in solid bar as example

–Explain the steps in 1D FE modeling

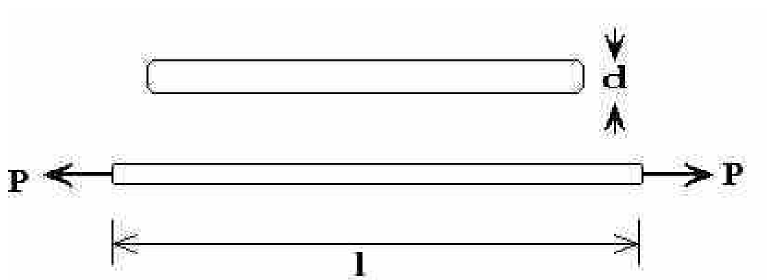
–Application to the steel plate problem

ONE-DIMENSIONAL MODEL

$u = u(x)$ displacement $\rightarrow \epsilon = \epsilon_{xx}(x) = \frac{du}{dx}$ strain

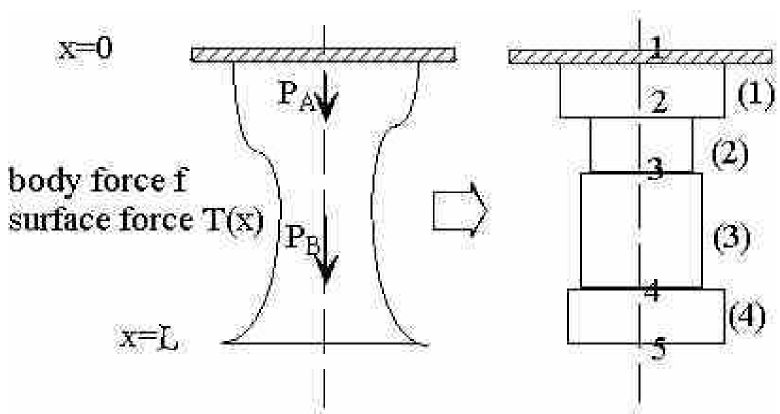
$\rightarrow \tau = \tau_{xx}(x)$ the only non-zero stress component (implicitly the Poisson ratio $\nu = 0$ or effects of transverse strain are neglected)

stress-strain relation $\tau = E\epsilon$



$$\frac{-\epsilon_{yy}}{\epsilon_{xx}} = \frac{\Delta d/d}{\Delta l/l} = \nu$$

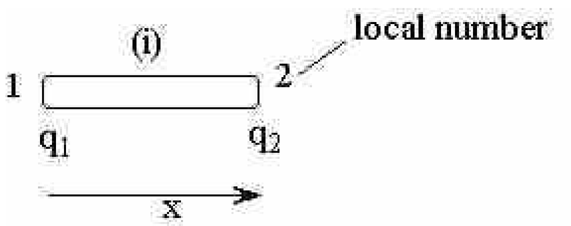
External loads:
$f = f(x)$ body force
$T = T(x)$ etc. surface traction – force per unit length in 1D



STEPS IN FINITE-ELEMENT MODELING

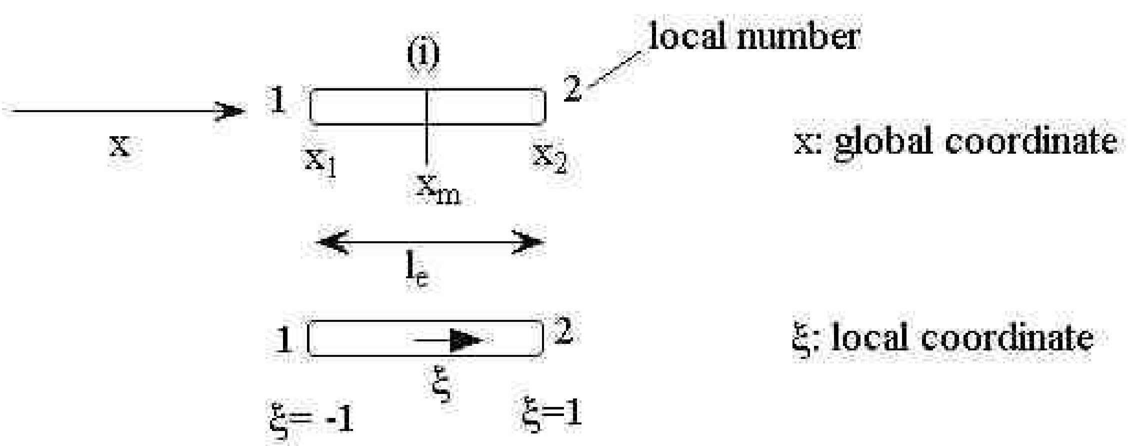
Step 1. element division: in 1D each element has a uniform cross section, uniform traction, uniform body force density. A, T, f may differ from element to element. It is convenient to define a node at each location where a point load is applied. Elements can have different lengths. Let q_1, q_2, q_3, q_4, q_5 be the displacements at node points. Then $Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}$ is the global displacement vector. Each node has one degree of freedom.

Element connectivity table



Elements	Nodes		Notes
e	1	2	local number
1	1	2	global number
2	2	3	
3	3	4	
4	4	5	

Step 2. Analysis within each local element



$$\xi \equiv \frac{2(x - x_1)}{x_2 - x_1} - 1$$

Establishing displacement field within an element using information at the nodes.

$$u(\xi) = N_1(\xi)q_1 + N_2(\xi)q_2$$

$N_1(\xi)$ and $N_2(\xi)$ are shape functions: first derivatives must be finite within each element, u must be continuous across the element boundary.

Example:

Linear interpolation

$$N_1(\xi) = \frac{1-\xi}{2} \Leftrightarrow N_1(-1) = 1,\ N_1(1) = 0$$

$$N_2(\xi) = \frac{1+\xi}{2} \Leftrightarrow N_2(-1) = 0,\ N_2(1) = 1$$

(satisfy both requirements)

Or $u = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \cdot (q_1,\, q_2) = \mathbf{N} \cdot \mathbf{q}$

$\mathbf{q}$ element displacement vector

Note also the relationship between global coordinate and local coordinate

$$x = N_1 x_1 + N_2 x_2$$

u and x are interpolated using same shape functions (isoparametric formulation).

Strain field in the element

$$\epsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{du}{d\xi}/\frac{dx}{d\xi} = \frac{q_2 - q_1}{x_2 - x_1} = \mathbf{B} \cdot \mathbf{q}$$

where $\mathbf{B} = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore,

$$\tau = E\mathbf{B} \cdot \mathbf{q}$$

Step 3. Apply approximation method, e.q. Galerkin approach

Governing eqn.

$$\frac{d\tau A}{dx} + fA + T + \sum P_i \delta(x - x_i) = 0$$

Introduce a test displacement field $\phi = \phi(x)$

$$\int_0^L \Big(\frac{d\tau A}{dx} + fA + T + \sum P_i \delta(x - x_i)\Big)\phi dx = 0$$

integration by parts

$$\underbrace{\int_0^L \tau \cdot \epsilon(\phi) A dx}_{\text{internal virtual work}} \quad \underbrace{- \int_0^L \phi \cdot fAdx - \int_0^L \phi T dx - \sum_i \phi_i P_i}_{\text{external virtual work}} = 0$$

Summation over elements

$$\sum_e \int_e \epsilon E \epsilon(\phi) A dx - \sum_e \int_e \phi \cdot fAdx - \sum_e \int_e \phi T dx - \sum_i \phi_i p_i = 0$$

ϵ–strain due to actual loads
$\epsilon(\phi)$–strain due to virtual displacement field
Discretization of $\phi(x)$: $(\psi_1,\ \psi_2,\ ...,\ \psi_5)^T$

If same interpolation scheme is used

$$\phi = \mathbf{N} \cdot \mathbf{\Psi}$$

$$\epsilon(\phi) = \mathbf{B} \cdot \mathbf{\Psi}$$

$$\begin{aligned}
\int_e \epsilon E \epsilon(\phi) A dx &= \int_e \mathbf{q} \cdot \mathbf{B}\, E\, \mathbf{B} \cdot \mathbf{\Psi} A dx \\
&= \mathbf{q}^T [\mathbf{B}^T B \frac{EAe}{2}(x_2 - x_1) \int_{-1}^{1} d\xi] \mathbf{\Psi} \\
&= \mathbf{q}^T K^e \mathbf{\Psi} \\
&= \mathbf{\Psi}^T K^e \mathbf{q} \\
&= (\psi_1,\, \psi_2) \; \underbrace{\frac{EAe}{x_2 - x_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{element stiffness matrix}} \; \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \text{ for each element}
\end{aligned}$$

where $l_e = x_2 - x_1$.

Similarly

$$\begin{aligned}
\int_e \phi f A dx &= \mathbf{\Psi}^T \mathbf{f}^e \\
&= (\psi_1,\, \psi_2) \frac{A_e(x_2 - x_1) f}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

where A_e is the cross section area, $l_e = x_2 - x_1$ is the element length.

$$\begin{aligned}
\int_e \phi T dx &= \mathbf{\Psi} \cdot \mathbf{T}^e \\
&= (\psi_1,\, \psi_2) \frac{T l_e}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

$\Longrightarrow \sum_e \psi^T k^e q - \sum_e \psi^T f^e - \sum_e \psi^T T^e - \sum_i \psi_i p_i = 0$

$\longrightarrow \Psi^T (KQ - F) = 0$

$\longrightarrow$ a linear system:

$$KQ = F$$

where

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{pmatrix} \tag{1}$$

$$K = E \begin{bmatrix} \frac{A_1}{l_1} & -\frac{A_1}{l_1} & 0 & 0 & 0 \\ -\frac{A_1}{l_1} & \frac{A_1}{l_1} + \frac{A_2}{l_2} & -\frac{A_2}{l_2} & 0 & 0 \\ 0 & -\frac{A_2}{l_2} & \frac{A_2}{l_2} + \frac{A_3}{l_3} & -\frac{A_3}{l_3} & 0 \\ 0 & 0 & -\frac{A_3}{l_3} & \frac{A_3}{l_3} + \frac{A_4}{l_4} & -\frac{A_4}{l_4} \\ 0 & 0 & 0 & -\frac{A_4}{l_4} & \frac{A_4}{l_4} \end{bmatrix} \tag{2}$$

global or structural stiffness matrix

$$F = \begin{bmatrix} \left(\frac{A_1 l_1 f}{2} + \frac{l_1 \tau_1}{2}\right) \\ \left(\frac{A_1 l_1 f}{2} + \frac{l_1 \tau_1}{2}\right) + \left(\frac{A_2 l_2 f}{2} + \frac{l_2 \tau_2}{2}\right) \\ \left(\frac{A_2 l_2 f}{2} + \frac{l_2 \tau_2}{2}\right) + \left(\frac{A_3 l_3 f}{2} + \frac{l_3 \tau_3}{2}\right) \\ \left(\frac{A_3 l_3 f}{2} + \frac{l_3 \tau_3}{2}\right) + \left(\frac{A_4 l_4 f}{2} + \frac{l_4 \tau_4}{2}\right) \\ \left(\frac{A_4 l_4 f}{2} + \frac{l_4 \tau_4}{2}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ p_a \\ 0 \\ p_b \\ 0 \end{bmatrix} \tag{3}$$

Note: the dimension of K is $N \times N$, N is the total degree of freedom; K is symmetric and banded matrix (the bandwidth depends on the numbering scheme for the nodes).

BC's can be used to reduce the DOF of the system.

Step 4. solve the matrix system by (direct) Gaussian elimination or iterative method (conjugate gradient method).

GALERKIN FINITE-ELEMENT METHOD APPLIED TO THE STEEL PLATE PROBLEM

Governing differential eqn

$$\sigma(x)A(x) = \sigma(x+dx)A(x+dx) + \rho g A(x)dx + \text{ external load in } x \leftrightarrow (x+dx)$$

Divide through by dx and let $dx \to 0$

$$\frac{d}{dx}(A\sigma) + \rho g A + \sum_i P_i \delta(x - x_i) = 0$$

note x_i location of external loads

$$\int_\Omega \delta(x - x_i)dx = \begin{cases} 1, \text{if } \Omega \text{ covers } x_i \\ 0, \text{if } \Omega \text{ does not covers } x_i \end{cases} \tag{4}$$

or

$$\frac{d}{dx}(A(x)E\frac{du}{dx})dx + \rho g A(x) + P\delta(x - 12) = 0 \tag{5}$$

BC's

$$\begin{aligned} u(x=0) &= 0 \\ \frac{du}{dx}(x=24) &= 0 \end{aligned}$$

Assume: $q(x)$ is approximate soln, $\phi(x)$ is weighting fcn.

$$\int_0^{24} \phi(x)\frac{d}{dx}(AE\frac{dq}{dx})dx + \int_0^{24} \phi\rho g A(x)dx + \phi(x=12)p = 0$$

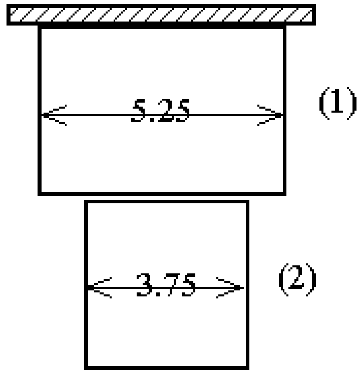
$$\phi(x)AE\frac{dq}{dx}|_{x=0}^{24} - \int_0^{24} AE\frac{dq}{dx}\frac{d\phi}{dx}dx + \int_0^{24} \phi\rho g A(x)dx + \phi(x=12)P = 0$$

Note that the exact BC's are used here for the first term. The first item is zero since $\phi(x=0) = 0$ and $\frac{dq}{dx}|_{x=24} = 0$.

$$-\int_0^{24} AE\frac{dq}{dx}\frac{d\phi}{dx}dx + \int_0^{24} \phi\rho g A(x)dx + \phi(x=12)P = 0 \tag{6}$$

This is the starting point for integration over elements.

Use two elements - Average cross-section areas:



$$
\begin{aligned}
-\int_0^{24} AE\frac{dq}{dx}\frac{d\phi}{dx}dx &= -\int_0^{12} A^{(1)}E\frac{q_1-q_0}{12}\frac{\phi_1-\phi_0}{12}dx - \int_{12}^{24} A^{(2)}E\frac{q_2-q_1}{12}\frac{\phi_2-\phi_1}{12}dx \\
&= -(\phi_1,\, \phi_2)\begin{bmatrix} \frac{5.25+3.75}{12} & -\frac{3.75}{12} \\ -\frac{3.75}{12} & \frac{3.75}{12} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} E
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{24} \phi\rho g A(x)dx + \phi(x=12)P &= \int_0^{12} \frac{x}{12}\phi_1 \cdot \rho g \cdot 5.25dx \\
&\quad + \int_{12}^{24} (\frac{24-x}{12}\phi_1 + \frac{x-12}{12}\phi_2)\rho g \cdot 3.75dx + P\phi_1 \\
&= (\phi_1,\, \phi_2)\begin{pmatrix} (31.5+22.5)\rho g + P \\ 22.5\rho g \end{pmatrix}
\end{aligned}
$$

$$
\Longrightarrow \frac{E}{12}\begin{bmatrix} 9 & -3.75 \\ -3.75 & 3.75 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 54\rho g + P \\ 22.5\rho g \end{bmatrix}
$$

$$
\Longrightarrow \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 9.2720e-6 \\ 9.9527e-6 \end{pmatrix}
$$

Exactly the same as what was obtained in our previous lecture and also as what was obtained in computer session 2 using ANSYS with 1D Link elements.

How about using the Principle of Minimum Potential Energy?

$$
\begin{aligned}
\Pi(q(x)) &= \frac{1}{2}\int \sigma\cdot\epsilon A dx - Pq_1 - \int \rho g q(x) A dx \\
&= \frac{1}{2}\int_0^{12} A^{(1)}E\frac{q_1}{12}\cdot\frac{q_1}{12}dx + \frac{1}{2}\int_{12}^{24} A^{(2)}E\frac{q_2-q_1}{12}\cdot\frac{q_2-q_1}{12}dx - Pq_1 - \int \rho g q(x) A dx \\
&= E\frac{q_1}{12}\cdot\frac{q_1}{12}5.25\frac{12}{2} + E\frac{(q_2-q_1)}{12}\cdot\frac{(q_2-q_1)}{12}3.75\frac{12}{2} \\
&\quad -Pq_1 - q_1(21.5+22.5)\rho g - q_2 22.5\rho g
\end{aligned}
$$

Setting $\partial\Pi(q(x))/\partial q_1 = 0$ and $\partial\Pi(q(x))/\partial q_2 = 0$ will lead to the exact same equations:

$$\frac{E}{12}\begin{bmatrix} 9 & -3.75 \\ -3.75 & 3.75 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 54\rho g + P \\ 22.5\rho g \end{bmatrix}$$

Use two elements - Variable cross-section areas: $A(x) = 6 - x/8$

$$\begin{aligned} -\int_0^{24} AE\frac{dq}{dx}\frac{d\phi}{dx}dx &= -\int_0^{12}(6-\frac{x}{8})E\frac{q_1-q_0}{12}\frac{\phi_1-\phi_0}{12}dx - \int_{12}^{24}(6-\frac{x}{8})E\frac{q_2-q_1}{12}\frac{\phi_2-\phi_1}{12}dx \\ &= -(\phi_1,\, \phi_2)\begin{bmatrix} \frac{7+5}{16} & -\frac{5}{16} \\ -\frac{5}{16} & \frac{5}{16} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} E \end{aligned}$$

This part is actually the same as what was obtained from the "average-area" treatment.

$$\begin{aligned} \int_0^{24}\phi\rho A(x)dx + \phi(x=12)P &= \int_0^{12}\frac{x}{12}\phi_1\cdot\rho\cdot(6-\frac{x}{8})dx \\ &+ \int_{12}^{24}(\frac{24-x}{12}\phi_1 + \frac{x-12}{12}\phi_2)\rho\cdot(6-\frac{x}{8})dx + P\phi_1 \\ &= (\phi_1,\, \phi_2)\begin{pmatrix} (30+24)\rho + P \\ 21\rho \end{pmatrix} \end{aligned}$$

$$\Longrightarrow \frac{E}{16}\begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 54\rho + P \\ 21\rho \end{bmatrix}$$

$$\Longrightarrow \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 9.2396e-6 \\ 9.8749e-6 \end{pmatrix}$$

This is exactly the same as what was obtained in computer session 2 using ANSYS with 2D solid elements (zero Poisson ratio).

The above solution procedure, is more logical than the previous approach using element-averaged properties. Although the solution is more accurate at $x = 24$, it is less accurate at $x = 12$.

Note that, regardless of which method being used to handle variable properties, the solution will converage to the exact solution as the number of elements increases.