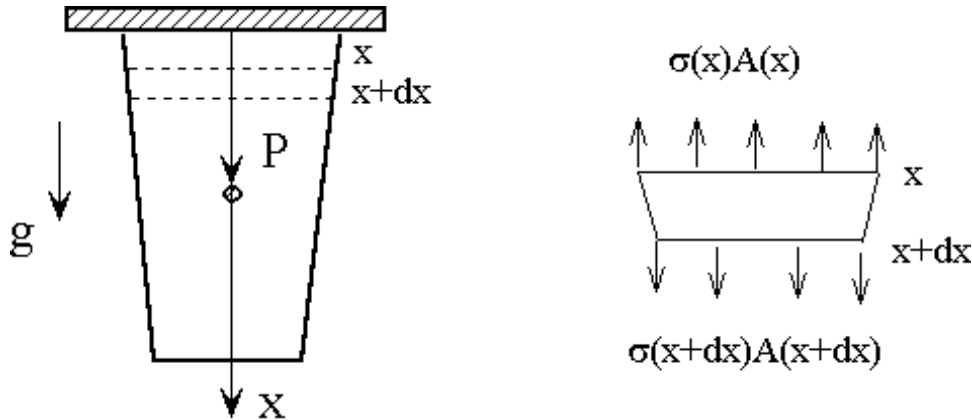


APPROXIMATION METHODS for STATIC PROBLEMS

Lecture Objectives:

- Differential vs integral formulations
- Variational approach
- Global approximation methods: (1) Rayleigh-Ritz method; (2) Galerkin's method

BECK TO THE STEEL PLATE PROBLEM



What is the governing differential equation for the 1D plate model?

If one takes a differential region from x to $x + dx$, force balance would give

$$\sigma(x)A(x) = \sigma(x + dx)A(x + dx) + \rho A(x)dx + P\Delta(x - 12 < x + dx).$$

where $\Delta(x - 12 < x + dx) = 1$ if $x - 12 < x + dx$ and zero otherwise.

Divide through by dx and let $dx \rightarrow 0$

$$\frac{d}{dx} \left(EA \frac{du}{dx} \right) + \rho A + P\delta(x - 12) = 0; \tag{1}$$

$$BC's : u(x = 0) = 0, \quad \frac{du}{dx}(x = 24) = 0. \tag{2}$$

This is the differential formulation.

Alternative formulation in terms of total potential energy Π :

$$\begin{aligned} \Pi &\equiv \text{Strain energy due to internal stress} - \text{work done by all external forces} \\ &= \frac{1}{2} \int_0^{24} \sigma \epsilon A dx - \left[\underbrace{\int_0^{24} u \rho A dx}_{\text{work done by gravity}} + \underbrace{Pu(x = 12)}_{\text{work done by P}} \right]. \end{aligned} \tag{3}$$

THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY: The equilibrium state corresponds to a state of minimum Π . OR of all the geometrically possible shapes that a system can assume, the true one, corresponding to the satisfaction of stable equilibrium of the system, is identified by a minimum value of the total potential energy.

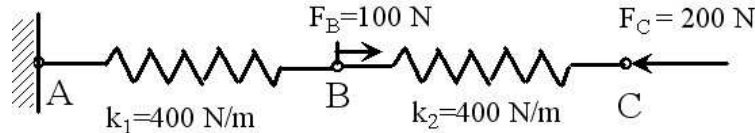
$$\Pi = \int_0^{x_c} k_1 x dx + \int_0^{x_c} k_2 x dx - F x_c$$

$$\Pi = \frac{1}{2}(k_1 + k_2)x_c^2 - F x_c$$

Setting $\partial\Pi/\partial x_c = 0$:

$$(k_1 + k_2)x_c - F = 0$$

Example 2:



Find the displacement at points B and C.

Method 1: Force balance method

$$k_1 x_b + k_2(x_b - x_c) = F_b$$

$$k_2(x_b - x_c) = F_c$$

Solve for x_b and x_c

$$x_b = -0.25m, \quad x_c = -0.75m$$

Method 2: Principle of minimum potential energy

$$\Pi = \frac{1}{2}k_1 x_b^2 + \frac{1}{2}k_2(x_b - x_c)^2 - F_b x_b + F_c x_c$$

Setting $\partial\Pi/\partial x_b = 0$ and $\partial\Pi/\partial x_c = 0$:

$$k_1 x_b + k_2(x_b - x_c) - F_b = 0$$

$$-k_2(x_b - x_c) + F_c = 0$$

RAYLEIGH-RITZ METHOD (one way to construct approximate soln to a problem)

This method consists of three steps:

Step 1: Construct an assumed displacement field using N independent, known basis functions

$$q(x) = \sum_{i=1}^N a_i \psi_i(x)$$

Here $\psi_i(x)$ are known functions, and $\psi_i(x)$ should satisfy so-called essential BC's, i.e., all BC's in terms of the primary variable (here the deformation) only, not its derivatives.

Step 2: Write $\Pi = \Pi(a_1, a_2, a_3, \dots, a_N)$.

Step 3: Set up a linear system by

$$\frac{\partial \Pi}{\partial a_i} = 0, \text{ for } i = 1, 2, 3, \dots, N$$

In the last step we optimize a_i so that the constructed function is close to the exact solution.

For example, in the steel plate problem, if we assume

$$u(x) \approx q(x) \equiv 48ax - ax^2,$$

where $q(x)$ satisfies both BC's. Then

$$\begin{aligned} \Pi(q(x)) &= \frac{1}{2} \int_0^{24} EA \left(\frac{dq}{dx} \right)^2 dx - \int_0^{24} \rho A q dx - Pq(x=12) \\ &= \frac{1}{2} \int_0^{24} E \left(6 - \frac{x}{8} \right) (48a - 2ax)^2 dx - \int_0^{24} \rho \left(6 - \frac{x}{8} \right) (48ax - ax^2) dx - 432aP \\ &= 50,688Ea^2 - 36,864\rho a - 432aP \end{aligned} \quad (7)$$

Then $d\Pi/da = 0$ gives

$$a = \frac{256\rho + 3P}{704E} = 1.7642e - 8.$$

Or approximate solutions:

$$\begin{aligned} u(x=12) &\approx q(x=12) = 7.6213e - 6 \\ u(x=24) &\approx q(x=24) = 10.1618e - 6 \end{aligned}$$

GALERKIN METHOD:

Again write the approximate solution as

$$q(x) = \sum_{i=1}^N a_i \psi_i(x)$$

Here all the basis functions $\psi_i(x)$ satisfy BC's.

Define the residual

$$\epsilon(x) \equiv \frac{d}{dx} \left(EA \frac{dq}{dx} \right) + \rho A + P\delta(x-12)$$

The coefficients a_i in the solution are determined by

$$\int_0^{24} \underbrace{\phi_i(x)}_{\text{Weighting functions}} \epsilon(x) dx = 0, \text{ for } i = 1, 2, \dots, N. \quad (8)$$

The weighting functions are constructed using the same basis functions ψ_i .

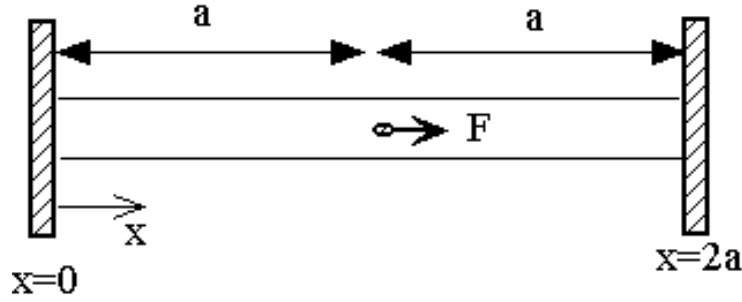
Eqn (8) can be written as

$$\begin{aligned} 0 &= \int \phi_i \frac{d}{dx} \left(EA \frac{dq}{dx} \right) + \int \rho A \phi_i dx + P\phi_i(x=12) \\ &= \underbrace{\phi_i EA \frac{dq}{dx}}_{=0, \text{ due to BC's}} - \int EA \frac{dq}{dx} \frac{d\phi_i}{dx} dx + \int \rho A \phi_i dx + P\phi_i(x=12) \end{aligned} \quad (9)$$

Eqn (9) is used as a starting point instead of Eqn (8).

ANOTHER EXAMPLE:

A linear elastic one-dimensional rod of length $2a$, body-force neglected. Elastic modulus E and cross-sectional area A are given. A point force F is applied at the center location, point 1. Find u_1 and stress in the rod.



Differential formulation:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + F\delta(x-a) = 0; \quad (10)$$

$$BC's: \quad u(x=0) = 0, \quad u(x=2a) = 0. \quad (11)$$

Exact solution is

$$u(x) = \begin{cases} \frac{x}{a}u_1, & \text{for } 0 \leq x \leq a; \\ u_1\frac{2a-x}{a}, & \text{for } a \leq x \leq 2a; \end{cases} \quad (12)$$

with

$$u_1 = \frac{aF}{2EA}$$

Now let us use the approximate methods. We shall use the following trial function:

$$q(x) = u_1\left[2\frac{x}{a} - \left(\frac{x}{a}\right)^2\right],$$

where u_1 represents the deformation at $x = a$. We note $q(x)$ satisfies BC's.

Rayleigh-Ritz method:

$$\begin{aligned} \Pi &= \frac{1}{2} \int_0^{2a} EA\left(\frac{dq}{dx}\right)^2 dx - Fq(x=a) \\ &= \frac{1}{2} \int_0^{2a} EA\left[\frac{2u_1}{a}\left(1 - \frac{x}{a}\right)\right]^2 dx - Fu_1 \\ &= \frac{4u_1^2 EA}{3a} - Fu_1. \end{aligned} \quad (13)$$

Now $\partial\Pi/\partial u_1 = 0$ gives

$$u_1 = \frac{3aF}{8EA}$$

Galerkin method:

$$q(x) = u_1 \left[2\frac{x}{a} - \left(\frac{x}{a}\right)^2 \right],$$

$$\phi(x) = \phi_1 \left[2\frac{x}{a} - \left(\frac{x}{a}\right)^2 \right]$$

The integral formulation is

$$-\int_0^{2a} EA \frac{dq}{dx} \frac{d\phi}{dx} dx + F\phi_1 = 0$$

$$-\int_0^{2a} EA \frac{2u_1}{a} \left[1 - \frac{x}{a} \right] \frac{2\phi_1}{a} \left[1 - \frac{x}{a} \right] dx + F\phi_1 = 0$$

Performing the integration,

$$\phi_1 \left[F_1 - \frac{8EA}{3a} u_1 \right] = 0.0$$

Since ϕ_1 is an arbitrary constant, then

$$u_1 = \frac{3aF}{8EA}$$

Same as the result based on Rayleigh-Ritz method.

