A CASE PROBLEM WITH TWO OBJECTIVES

–The case problem: Deformation of steel plate under a point load and its own weight

–First objective: illustrate the steps involved in system analysis/model development/model solution/model results and evaluation

–Second objective: illustrate the finite element method

The Problem

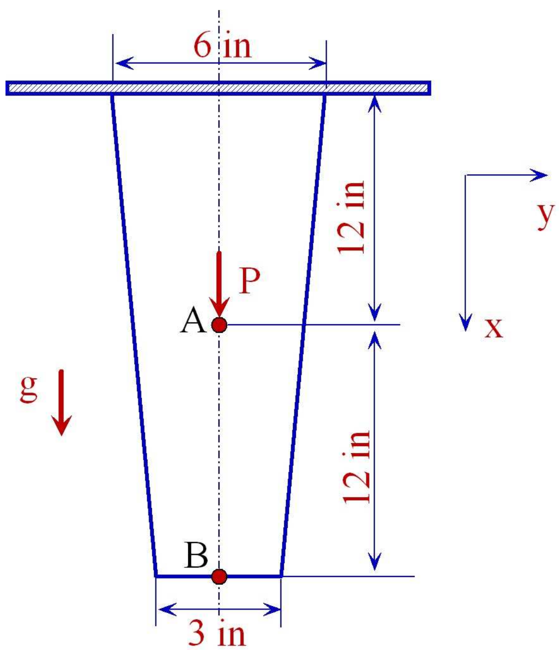
Consider a thin steel plate of uniform thickness of 1 in, subject to its weight and an external load $P = 100$ lbf applied at point A.

Weight density $\rho g = 0.2836$ lbf/in^3 and Young's modulus $E = 30 \times 10^6$ psi (lbf/in^2).

The unit system: [force] = lbf; [length]=in; [mass]=lbm.

Note the conversions: $g = 9.8\ m/s^2 = 9.8\ N/kg = 9.8 \times 0.224809\ lbf/(2.2046\ lbm) = 1.0\ lbf/lbm$

Then density $\rho = 0.2836\ lbm/in^3$, $E = 30 \times 10^6\ lbf/in^2$.



Find: deformation (or displacement) at points A and B.

MODEL DEVELOPMENT STEP:

Engineering assumptions:

(1) one dimensional model, deformation $u(x, y, z)$ is mainly a function of x:

$$u(x, y, z) \Rightarrow u(x)$$

deformation is a function of x only – known as "truss".

(2) Static problem

Mathematical formulation: setting up coordinates; introducing notation

Let $\sigma(x)$ be the normal stress at location x,

$u(x)$ be the deformation at location x,

$\epsilon(x) = \frac{du}{dx}$ the strain.

Physical principles:

Force balance (a special case of Newton's 2nd law)

$\sigma(x) = E\epsilon(x) = E\frac{du}{dx}$, Hooke's law.

Now to complete the formulation:

For the upper half $0 \leq x \leq 12$ in, force balance is

$$\underbrace{(6 - \frac{x}{8}) \times 1}_{\text{cross-section area}} \quad \times \quad \underbrace{E\frac{du}{dx}}_{\text{force/area}} \quad = \quad \underbrace{P}_{\text{point force}} \quad \underbrace{+\rho g \times (24 - x) \times 1 \times \frac{3 + (6 - \frac{x}{8})}{2}}_{\text{weight}} \tag{1}$$

For the lower half $12 \leq x \leq 24$ in, force balance is

$$(6 - \frac{x}{8}) \times 1 \times E\frac{du}{dx} = \rho g \times (24 - x) \times 1 \times \frac{9 - \frac{x}{8}}{2} \tag{2}$$

Boundary condition:

$$u(x = 0) = 0 \tag{3}$$

MODEL SOLUTION (I):

Analytical solution (knowledge of calculus etc.)

Integrate (1) from $x = 0$ to $x \leq 12$ with B.C. (3), to obtain

$$E\ u(x) = 8P \ln \frac{48}{48 - x} + \rho g \left[24x - \frac{x^2}{4} + 288 \ln \left(\frac{48 - x}{48}\right)\right] \tag{4}$$

Therefore, $u(x = 12) = 0.92705 \times 10^{-5} = 9.2705E - 6$ in

Integrate (2) from $x = 12$ to $x \leq 24$:

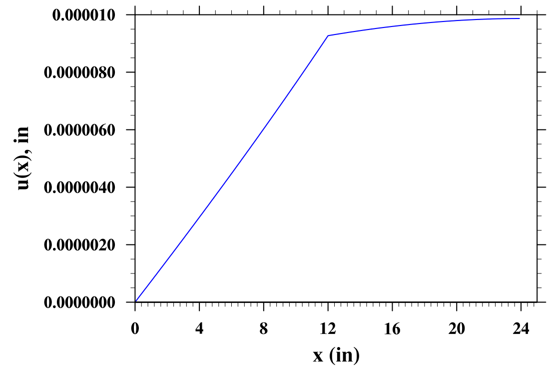
$$E[u(x) - u(x = 12)] = \rho g \left[24(x - 12) - \frac{x^2 - 144}{4} + 288 \ln \left(\frac{48 - x}{36}\right)\right] \tag{5}$$

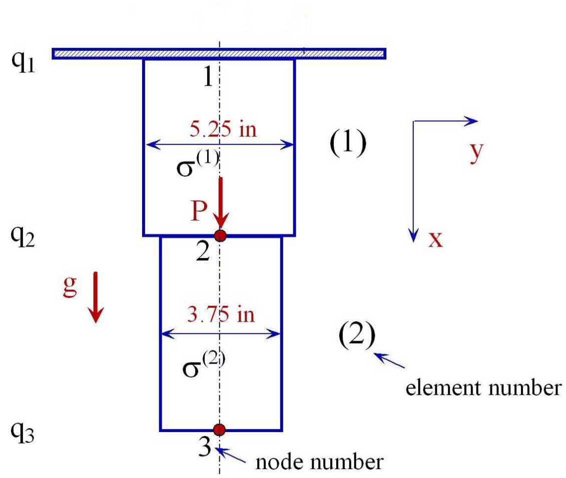
Therefore, $u(x = 24) = 9.8682E - 6$ in.

MODEL SOLUTION (II):

Finite element solution:

First set up discrete-element model (numerical model) use 2 elements, let the finite element solution of deformations at points 1, 2, 3 be q_1, q_2, and q_3.





Under 1D elements, we average out properties for each elements.

Use linear element: u is a linear function of x in each element.

$$u(x) = \frac{x}{12} q_2, \text{ for } 0 \le x \le 12$$

$$u(x) = \frac{24 - x}{12} q_2 + \frac{x - 12}{12} q_3, \text{ for } 12 \le x \le 24$$

The stress in each element:

$$\sigma = E \frac{du}{dx} = \left\{ \begin{array}{lll} \frac{E}{12} q_2 & \rightarrow & \sigma^{(1)} \\ \frac{E}{12}(q_3 - q_2) & \rightarrow & \sigma^{(2)} \end{array} \right\rangle \text{ stress at midpoint of elements}$$

Now employ force balance:

$$3.75 \times 1 \times \sigma^{(2)} = 6 \times 1 \times \rho g \times 3.75$$

or

$$\frac{E}{12}(q_3 - q_2) = 6 \rho g \tag{6}$$

$$5.25 \sigma^{(1)} = 12 \times 3.75 \rho g + 5.25 \times 6 \rho g + 100$$

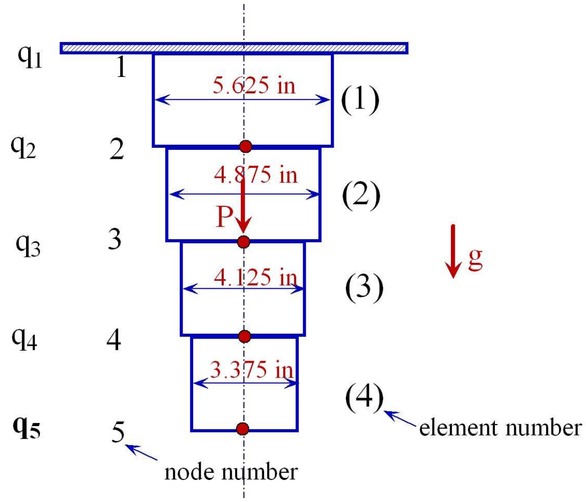
$$5.25 \frac{E}{12} q_2 = 121.6954 \tag{7}$$

(4)+(5) lead to a linear system:

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$$

$$\Rightarrow \begin{cases} q_2 = 9.27203E-6 \text{ in} \\ q_3 = 9.95267E-6 \text{ in} \end{cases}$$

More elements? (4 elements)



$$\begin{cases} 5.625E\frac{q_2}{6} = 100 + 91.125\rho g & \rightarrow \quad q_2 = 4.4744196E-6 \\ 4.875E\frac{q_3-q_2}{6} = 100 + 59.625\rho g & \rightarrow \quad q_3 = 9.2707129E-6 \\ 4.125E\frac{q_4-q_3}{6} = 32.625\rho g & \rightarrow \quad q_4 = 9.7193165E-6 \\ 3.375E\frac{q_5-q_4}{6} = 10.125\rho g & \rightarrow \quad q_5 = 9.8894765E-6 \end{cases}$$

Comparison	$u_A \times 10^6$ (in)	$u_B \times 10^6$ (in)
Exact 1D	9.2705	9.8682
2-element	9.27203	9.95267
4-element	9.27071	9.88948

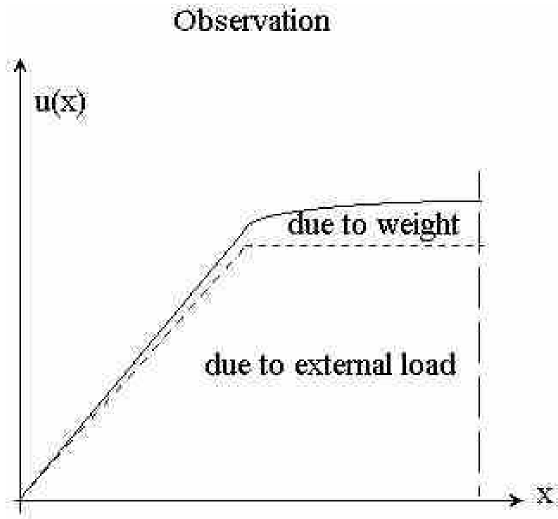
Soln improves as the number of elements increases.

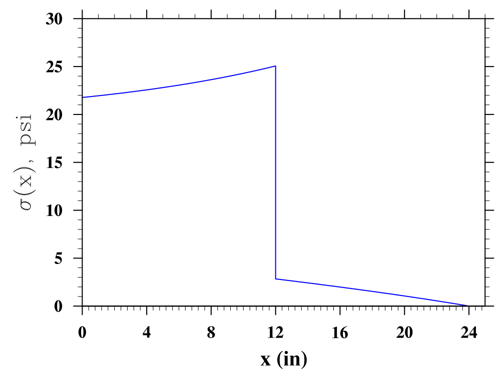
RESULTS AND MODEL EVALUATION

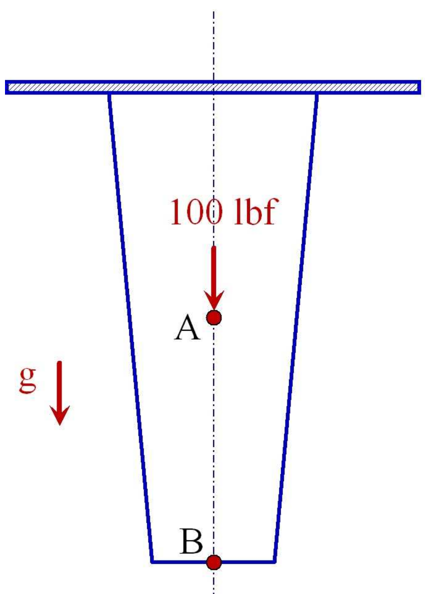
Where does the maximum stress occur?

What is the maximum stress?

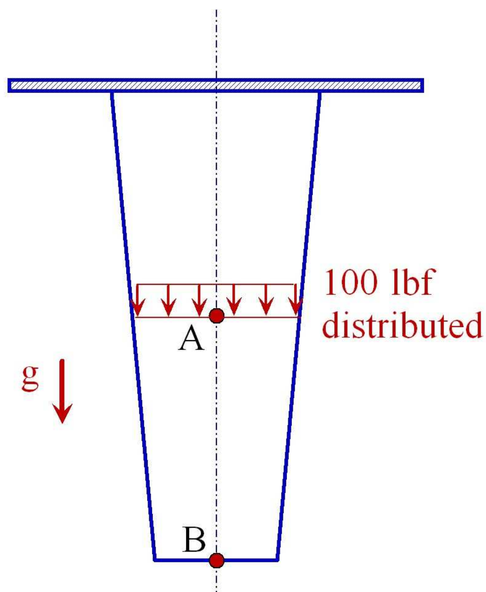
Is one-dimensional model adequate?







Probably NO



Probably YES
(better design
for small σ_{max})