MEEG 630, Intermediate Fluid Mechanics

Homework Set #1

1. Rewrite the following equation using tensor notation:

$$\rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla P + \mu\nabla^2\vec{u},$$

where $\vec{u}$ is the velocity vector, P is pressure, ρ is fluid density, and μ is viscosity.

2. Prove the following vector identity using tensor notation:

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}\nabla\cdot\vec{B} + (\vec{B}\cdot\nabla)\vec{A} - \vec{B}\nabla\cdot\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$$

3. Show that

$$C_{ik}C_{jk} = C_{ki}C_{kj} = \delta_{ij}$$

where C_{ij} is the direction cosine matrix for coordinate transformation and δ_{ij} is the Kronecker delta matrix. Any matrix obeying such a relationship is called an orthogonal matrix, because it represents transformation of one set of orthogonal axes into another.

4. Show that, for a second order tensor A, the following two quantities are invariant under the rotation of axes

$$I_1 = A_{ii}$$

$$I_2 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix}$$

Also verify that $I_2 = \frac{1}{2}\left[I_1^2 - A_{ij}A_{ji}\right]$.

5. The velocity components in an unsteady plane flow are given by

$$u = x + t \quad \text{and} \quad v = -y.$$

   (a) Determine the fluid acceleration at the point $x = 0$ and $y = 1$;
   (b) Sketch the streamlines passing through $x = 0$ and $y = 1$ at $t = 1$ and $t = 2$, respectively;
   (c) Determine the path line for the fluid particle located at the point $x = 0$ and $y = 1$ at $t = 0$.
   (d) If we slowly release some non-diffusive dye into the flow at the point $x = 0$ and $y = 1$, starting from $t = 0$, how would you describe the trace that this dye will make in the flow?

6. The velocity field of a certain flow is given by

$$u = 2xy^2 + 2xz^2, \qquad v = x^2 y, \qquad w = x^2 z$$

Consider the fluid region inside a spherical volume $x^2 + y^2 + z^2 = a^2$. Verify the validity of Gauss's theorem

$$\int \nabla\cdot\vec{u}\; dV = \int \vec{u}\cdot d\vec{A}$$

by integrating over the sphere. (You may introduce spherical coordinates such that $x = r\cos\theta$, $y = r\sin\theta\cos\phi$ and $z = r\sin\theta\sin\phi$. Also note that $\vec{u}\cdot d\vec{A} = u_r dA$ with $u_r = xu/r + yv/r + zw/r$.)