The Material Derivative

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Physics, both classical and quantum, is concerned with the motion of particles. In the classical domain, Newton's laws are stated in terms of forces acting on reactions of "bodies." Similarly, quantum mechanics is concerned with probabilities of finding particles at certain positions.

Direct application of these principles in fluid mechanics is not as simple as it is in a quantum mechanical experiment. After all, in the ocean, there are an infinite number of particles, and generally, there is no simple way to distinguish one from another. Worse, there is no unique way to define what makes up a particle in the ocean. Is it a water molecule? Even if the technology exists to follow individual molecules, this approach is doomed to failure since water is highly ionized, and the hydrogen and oxygen atoms that make up a water molecule, exist as a molecular unit for only a fraction of a second.

Perhaps an "effective" water particle could be defined by some correlation scale based on statistical molecular concepts? We all await this development. In the meantime, we will have to plug along with vague and nebulous concepts of a fluid particle, so that we can apply Newton's laws to fluid motion.

Here, in a nutshell, is the quandry. The theoretical paradigm, Newton's laws and the conservation of mass, apply to particles. But fluid mechanicists don't have access to individual particles; they have observations at geographic positions or calculations pertaining to specific nodes in a computational mesh. How to relate the two?

Fortunately, first year calculus (devised by Newton and Leibnitz) provides the bridge. Consider a property G associated with a generic particle. This could be temperature, density, velocity, whatever. Of course, G will vary from particle to particle and perhaps also in time because of unnamed exterior processes. Two mathematical descriptions of G are available. The fundamental description is particle based; i.e.,

$$G = G(X_i,\ t) \tag{1}$$

This means we associate G with specific water particles that are identified by their starting positions. As discussed above, we usually have to sit at a position and observe G. This gives

$$G = G(x_i,\ t) \tag{2}$$

Here G is associated with particles that are passing a specific geographic position.

These two descriptions are equivalent if we recognize that x_i is just the present position of the particle that started at X_i at $t = t_0$.

In other words

$$\begin{aligned} x_i &= x_i(X_i,\ t) \\ X_i &= X_i(x_i,\ t) \\ x_i(t = t_0) &= X_i \end{aligned} \tag{3}$$

Now the rate of change of G as seen by an observer moving with the particle that started at X_i is $\partial G/\partial t\ |_{X_i}$. Similarly, the rate of change of G as seen by an observer at position x_i is $\partial G/\partial t\ |_{x_i}$. Can the observations of the two observers be reconciled?

Yes, and you actually knew how to find the relation in your first calculus course. Since

$$\begin{aligned} G = G(X,\ t) &= G(x_i(X_j,\ t), t) \\ &= G(x_i,\ t) \end{aligned} \tag{4}$$

Then

$$\left(\frac{\partial G}{\partial t}\right)_{X_i} = \left(\frac{\partial G}{\partial t}\right)_{x_i} + \frac{\partial G}{\partial x_j}\frac{\partial x_j}{\partial t}. \tag{5}$$

Since $\partial x_i/\partial t = v_i$, (5) simplifies to

$$\left(\frac{\partial G}{\partial t}\right)_{X_i} = \left(\frac{\partial G}{\partial t}\right)_{x_i} + v_j\frac{\partial G}{\partial x_j}, \tag{6}$$

This equation states that the time rate of change of G seen by an observer moving with a particle (the most basic description) is the same as the time rate of change seen by an observer at the fixed position that the particle is passing by the time t plus the rate that G is advected by x_i.

Since $(\partial G/\partial t)_{X_i}$ is fundamental, it is given the special symbol DG/Dt. This is often referred to as the Lagrangian derivative, and descriptions based on particle historically are called Lagrangian, although, Lagrange probably had nothing to do with this. The position based description is called Eulerian.

The Lagrangian/Eulerian equivalence as expressed by

$$\frac{DG}{Dt} = \frac{\partial G}{\partial t} + v_j\frac{\partial G}{\partial x_j} \tag{7}$$

is as fundamental in fluid mechanics and classical physics as the wave-particle duality is in quantum mechanics.