deviate from Gaussianity and we expect such deviation to be of significance near the origin.

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Chaotic dynamics of heavy particle dispersion: Fractal dimension versus dispersion coefficients

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The chaotic dynamics of Lagrangian motion of particles in a steady Arnold–Beltrami–Childress (ABC) flow and a pseudoturbulence are investigated and the Lyapunov exponents and fractal dimensions of particle trajectories for different particle inertia and particle drift velocity are computed. The dispersion process of particles could be characterized by the fractal dimension and dispersion coefficients. The interesting behavior of fractal dimension of particle motion in ABC flow suggested the similarity of particle motion in ABC flow and in a mixing layer. The relationship between particle dispersion coefficient and fractal dimension was nearly linear for the pseudoturbulence.

Recently, methods of chaotic dynamics have been used to help understand particle transport, mixing, and dispersion. These methods provide information about the Lagrangian motion in the flow. Even in simple steady flow, the Lagrangian motion of fluid particles can be chaotic.[1-3] The Lagrangian motion of particles in the Arnold–Beltrami–Childress (ABC) flow was investigated recently by McLaughlin.[4] In this simple cellular flow, McLaughlin found that chaotic behavior of the particle trajectory existed and tended to be eliminated by the particle inertia and virtual mass; however, he did not quantify the chaotic behavior. Aref and Jones[5] found that if the flow exhibited chaotic advection, separation of diffusing fluid particles in a Stokes flow was enhanced.

The purpose of our study was to look for a possible relationship between the chaotic behavior of the system and particle dispersion in complex flows. Specifically, we compared the dispersion coefficients of heavy particles to the fractal dimension of the particle attractors. The dispersion coefficient ϵ describes the average rate of long-time dispersion of particles from a point source. It is also a measure of the rate of mean-squared separation of particles that start in the same neighborhood. The fractal dimension d_f is a measure of the phase-space volume explored by an orbit on the particle attractor. In this study we have computed d_f from the Lyapunov exponents (Kaplan–Yorke conjecture[6]), which measure the average rate of divergence (or convergence) of neighboring phase-space orbits. In the context of Kaplan and Yorke,[6] the fractal dimension is directly related to the spreading of neighboring orbits in phase space. This is not the same as the separation of particles in physical space. (Furthermore, ϵ and d_f are different in the sense that the dispersion coefficients are related to random processes while the fractal dimension of the systems considered here is a property related to deterministic dynamics.) We speculate, however, that particle dispersion is directly related to the Lyapunov spectrum, and we use the fractal dimension d_f as the most obvious measure of this spectrum. Thus we expect that the dispersion coefficient ϵ and the fractal dimension d_f, each suitably normalized, to be monotonically related. Our interest was to explore the degree of quantitative correlation.

We consider the motion of a heavy particle under Stokes drag in a flow field and assume the pressure gradient force, the virtual mass, and the Basset force can be neglected because of the large difference between the density of the heavy particle and the density of fluid. The equations of motion of a

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particle can be written in dimensionless form as

$$\frac{du_{pi}}{dt} = \frac{u_i(\mathbf{x}_P,t) - u_{pi}}{\mathrm{St}} - \frac{\gamma\delta_{i3}}{\mathrm{St}}, \quad i = 1,2,3; \tag{1a}$$

$$\frac{dx_{pi}}{dt} = u_{pi}, \quad i = 1,2,3. \tag{1b}$$

Here Stokes number, $\mathrm{St} \equiv \tau_a/T$, is the ratio of a particle's aerodynamic response time to a characteristic time of the flow, $\gamma = \tau_a g/v_0$ is the ratio of the particle's drift velocity $\tau_a g$ to a characteristic velocity of the flow, where g is the body force per unit mass, and $u_i(\mathbf{x},t)$ is a given nonlinear function of space and time representing the instantaneous fluid velocity. Therefore the motion of a heavy particle is controlled by both deterministic parameters, such as particle inertia, relative drift due to the constant body force acting on the particle, and the complex, nonlinear (possibly random) input of the fluid velocity. The dynamical system (1a) and (1b) is six dimensional, strongly nonlinear, and nonautonomous. It is also dissipative since the volume expansion rate associated with (1a) and (1b) is $-3/\mathrm{St}$ if the flow is incompressible.

The motion of a fluid particle is given by

$$\frac{dx_i(t)}{dt} = u_i(\mathbf{x},t), \quad i = 1,2,3, \tag{2}$$

which is a three-dimensional, nonlinear, volume preserving system. The dispersion coefficient and fractal dimension of a fluid particle will be used to nondimensionalize the motion of heavy particles.

Numerical calculations of Lyapunov exponents and fractal dimension of heavy particle motion [(1a) and (1b)] were carried out using the procedure of Wolf *et al.*[7] A base trajectory was produced first by numerical integration of the motion equations of a particle. Then, six neighboring trajectories were computed using the linearized version of the equations of motion, starting from six orthonormal directions. Gram–Schmidt reorthonormalization procedure on the vector frame was used after every time duration τ to form a new orthonormal set. After n intervals, the Lyapunov exponents were estimated by

$$\sigma_i(n) = \frac{1}{n\tau} \sum_{j=1}^{j=n} \log_2[\mathrm{Norm}_i(j)], \quad i = 1,\ldots,6, \tag{3}$$

where $\mathrm{Norm}_i(j)$ is the length of projection of the evolved vector i in its new normalization direction at time $j\tau$. If n is sufficiently large, σ_i will no longer depend on n. For all the results presented, n is 1000 and τ is 1.0. The Kaplan and Yorke or Lyapunov dimension,[6,7] d_f, was then calculated as

$$d_f = j + 1 + \sum_{i=1}^{j} \frac{\sigma_i}{|\sigma_{j+1}|}, \tag{4}$$

where j is defined by the condition that

$$\sum_{i=1}^{j} \sigma_i > 0 \quad \text{and} \quad \sum_{i=1}^{j+1} \sigma_i < 0. \tag{5}$$

However, if the fluid velocity, $u_i(\mathbf{x},t)$, was only a function of space coordinates, the system would be autonomous and the 1 in the rhs of Eq. (4) should be eliminated.

Two different flow fields, ABC flow and pseudoturbulence, which were used in this study, are described below.

ABC flow. The ABC flow, the same as used by McLaughlin,[4] has the following form:

$$u_1 = A\sin(2\pi x_3) + C\cos(2\pi x_2), \tag{6a}$$

$$u_2 = B\sin(2\pi x_1) + A\cos(2\pi x_3), \tag{6b}$$

$$u_3 = C\sin(2\pi x_2) + B\cos(2\pi x_1). \tag{6c}$$

This flow is an exact solution of Euler's equation because the vorticity vector is parallel to the velocity vector at all points in space. In the real world, this flow would decay because of the viscosity of the fluid. The details of the flow structure were discussed in Dombre *et al.*[3] In our study only one set of parameters, $A = 1$, $B^2 = 0.9$, $C^2 = 0.5$, was used.

The Lyapunov exponents for a Lagrangian fluid particle in this ABC flow were calculated. They are $\sigma_1 = 0.52$, $\sigma_2 = 0.00$, and $\sigma_3 = -0.52$. The sum of Lyapunov exponents is zero, which is expected for a volume preserving autonomous system.

Pseudoturbulence. The stochastic model for a Gaussian random velocity field proposed by Kraichnan[8] was used to simulate stationary, homogeneous turbulence.[9,10] This flow field is represented by the following equation, which is a linear superposition of a large number of Fourier modes with random amplitudes and phases:

$$\frac{u_i(\mathbf{x},t)}{v_0} = \sum_{n=1}^{N} [b_i^{(n)}\cos(\mathbf{k}^{(n)}\cdot\mathbf{x} + \omega^{(n)}t) + c_i^{(n)}\sin(\mathbf{k}^{(n)}\cdot\mathbf{x} + \omega^{(n)}t)]. \tag{7}$$

This equation is equivalent to the discretization of a Fourier transform over space and time for a velocity field. Here N is the number of modes and v_0 is the rms fluctuation velocity. The mean velocity is zero and the velocity field is understood to represent the velocity field in a frame of reference moving with the mean velocity of the flow. With proper filtering, the overall flow field is incompressible. The velocity correlation and wave number space spectrum used were the same as used by Wang and Stock.[10]

In the numerical calculation of Lyapunov exponents one realization of the coefficients for the flow given by Eq. (7) was used. All the coefficients, c_i and b_i, were held constant during the numerical integration of the particle's trajectory; thus the flow was deterministic. On the other hand, the dispersion coefficient was calculated from many realizations of the flow field using a different set of coefficients for each calculation.[10] In all the calculations N was set equal to 200.

In Fig. 1, the fractal dimension of the attractor of a heavy particle motion is shown as a function of particle inertia in the ABC flow for the case of zero drift velocity. The fractal dimension increases with St when St is less than one, and decreases with St when St is larger than one. The dispersion coefficient of particles in a plane mixing layer[11] has a similar behavior. In the ABC flow, the particle motion is controlled by the principal vortices,[3] while in the mixing layer it is controlled by the organized vortex structure. Consequently, both the fractal dimension and dispersion coefficients reach a maximum value when St is one. This implies that the fractal dimension and dispersion coefficients are directly related.

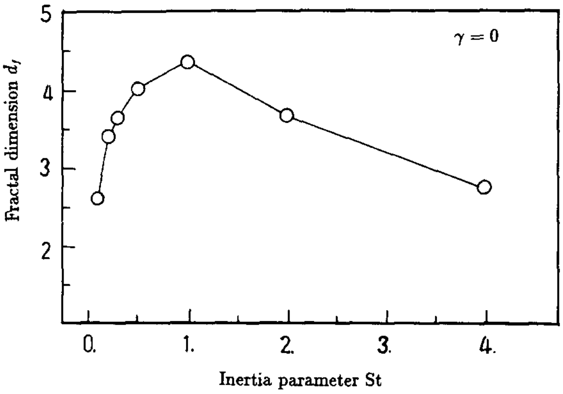

FIG. 1. Fractal dimension of particle motion in ABC flow as a function of particle inertia for zero drift velocity.

Figure 2 shows how the fractal dimension of the attractor of heavy particles in ABC flow changes with the drift parameter γ, with St = 1.0. For very small γ, the fractal dimension is almost constant. Then it decreases quickly with increasing γ for $0.3 < \gamma < 0.7$. For $\gamma > 0.7$, two peaks were found at γ close to 0.9 and 1.0. The Poincaré maps were plotted (but not included in this Letter) and they show that the larger the fractal dimension, the more the attractor spreads on the Poincaré map. At the peak points of the jumps, the particle was *suspended* in the flow and possessed a unique structure. For $\gamma > 1.0$, the fractal dimension was almost constant.

Figures 3 and 4 show the normalized fractal dimensions and dispersion coefficients for a particle moving in the pseudoturbulence. The dispersion coefficient was normalized by its value for a fluid particle, which was $0.72 T_{mE} v_0$ where T_{mE} is the integral time scale of the flow. The fractal dimension was also nondimensionalized by a value of 4, which represents the dimension for the motion of a fluid particle in an unsteady flow. The fractal dimension and dispersion coefficients increase with particle inertia for $\gamma = 1.0$. Furthermore, the relation between dispersion coefficient and fractal dimension was *linear*.

Figures 5 and 6 show the normalized dispersion coeffi-

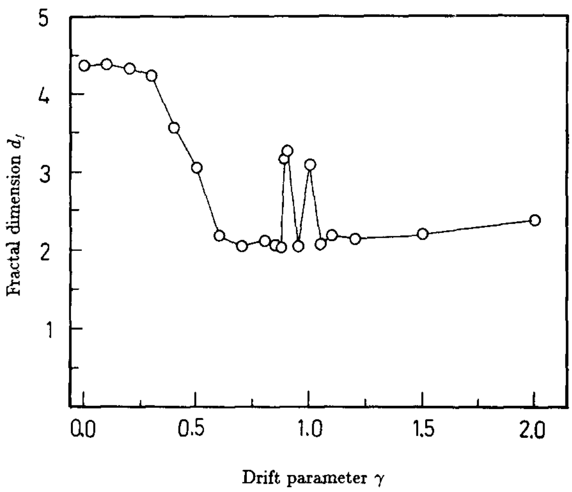

FIG. 2. Fractal dimension of particle motion in ABC flow as a function of drift parameter γ when inertia parameter St = 1.0.

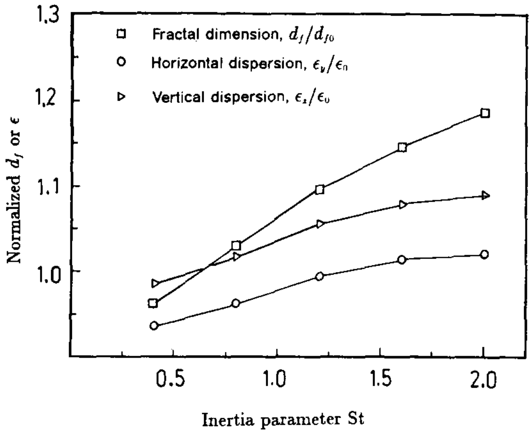

FIG. 3. Fractal dimension and dispersion in pseudoturbulence as a function of particle inertia with $\gamma = 1.0$.

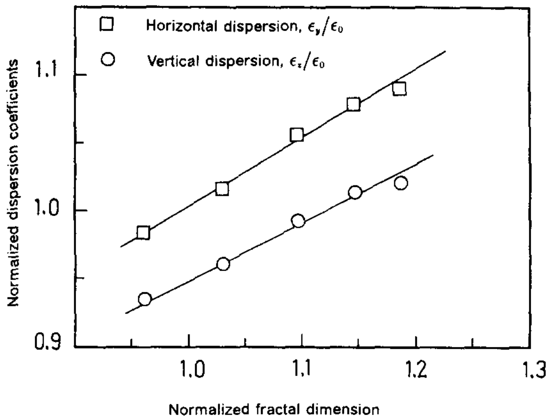

FIG. 4. Dispersion coefficients as a function of fractal dimension when drift parameter is fixed.

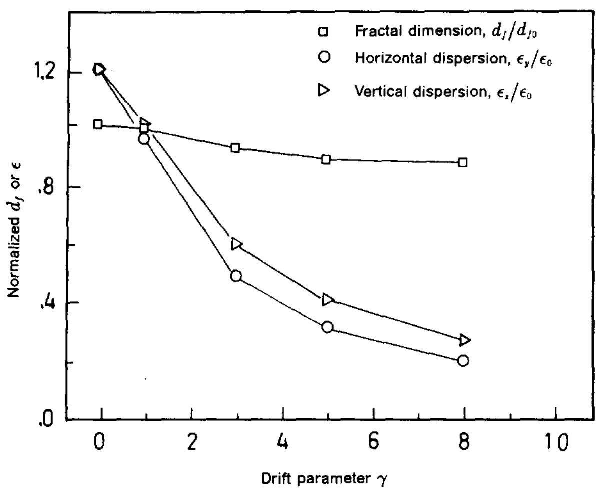

FIG. 5. Fractal dimension and dispersion coefficients in pseudoturbulence as a function of drift parameter γ with St = 0.8.

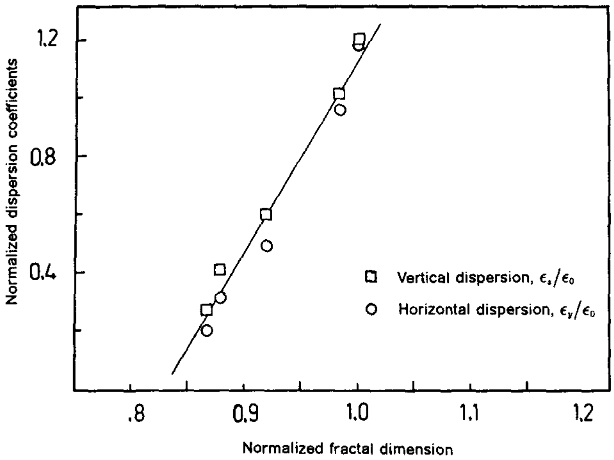

FIG. 6. Dispersion coefficients as a function of fractal dimension when inertia parameter is fixed.

cients and fractal dimensions for St $= 0.8$ with various γ in the pseudoturbulence. The fractal dimension and dispersion coefficients decrease with increasing γ as expected. Again, the relation between the fractal dimension and dispersion coefficients was *linear.*

The results of the study lead us to the following conclusions: (1) fractal dimension can be used to characterize the dispersion of particles; (2) the dispersion of particles in ABC flow is similar to the dispersion process of particles in a mixing layer; (3) for isotropic, homogeneous pseudoturbulence, the relation between the fractal dimension and dispersion coefficient is linear; and (4) the fractal dimension and Lyapunov spectrum possess *physical* meaning not only for a deterministic system with deterministic controls, but also for a deterministic system with both deterministic and random controls.

The present numerical results are worth analytical studies. The physical meaning of fractal dimension for a system with random excitation should be explored. Rigorous analysis to relate fractal dimension to dispersion coefficients should be attempted.

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