Is the Kolmogorov Refined Similarity Relation Dynamic or Kinematic?

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High-resolution Navier-Stokes simulations are used to test the Kolmogorov refined similarity hypothesis (RSH). It is found that the statistical dependence between velocity increments and dissipation rate invoked by RSH is mostly due to kinematics at moderate Reynolds numbers. However, a dynamically induced contribution can be demonstrated. The longitudinal velocity increment is correlated more strongly with the purely longitudinal component of dissipation rate than with other components. An alternative formulation of the RSH is offered.

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In order to take account of spatial fluctuations of dissipation rate in fluid turbulence, Kolmogorov [1] made a revision (K62) of his 1941 (K41) theory [2] by introducing the *refined similarity hypothesis* (RSH), which relates velocity difference δu_r across a distance r and the locally averaged dissipation rate $\epsilon_r(\mathbf{x})$. Let $\mathbf{x} = (x, y, z)$, $\mathbf{u} = (u, v, w)$, and let $\mathbf{r}$ point along the x axis. Then $\delta u_r \equiv |\Delta u_r| \equiv |u(\mathbf{x} + \mathbf{r}/2) - u(\mathbf{x} - \mathbf{r}/2)|$, while $\epsilon_r(\mathbf{x})$ is defined as the average of the dissipation at a point $\epsilon(\mathbf{x})$ over a domain $D(\mathbf{x}, \mathbf{r})$ about $\mathbf{x}$. Here $\epsilon(\mathbf{x}) = 2\nu s_{ij}s_{ji}$, ν is the kinematic viscosity, and $s_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the strain tensor. The averaging domain $D(\mathbf{x}, \mathbf{r})$ is conventionally a sphere centered on $\mathbf{x}$, but in experiments, most numerical investigations, and the present Letter, it is the line segment $(\mathbf{x} - \mathbf{r}/2, \mathbf{x} + \mathbf{r}/2)$.

Kolmogorov hypothesized that at high Reynolds numbers the conditional moments of δu_r are functions only of ν, ϵ_r, and r:

$$\langle(\delta u_r)^n|\epsilon_r\rangle = (r\epsilon_r)^{n/3} g_n(R_{\epsilon_r}), \tag{1}$$

for $r \ll L$, where L is the integral scale, $\langle\cdot|\epsilon_r\rangle$ denotes an ensemble average conditioned on a given value ϵ_r, and the g_n are functions only of local Reynolds number $R_{\epsilon_r} = \epsilon_r^{1/3} r^{4/3}/\nu$. Kolmogorov further proposed that if $R_{\epsilon_r} \gg 1$, there is an inertial range of r in which

$$\langle(\delta u_r)^n|\epsilon_r\rangle = D_n(r\epsilon_r)^{n/3}, \tag{2}$$

where the D_n are constants. Equation (2) follows if

$$\delta u_r = \beta(r\epsilon_r)^{1/3}, \tag{3}$$

where β is a stochastic variable independent of r and statistically independent of ϵ_r.

The RSH combines three physical assumptions: first, that the NS dynamics induces a statistical dependence between velocity fluctuations over spatial scales $\gg \ell_d$ and dissipation averaged over such scales; second, that this dependence persists as Taylor microscale Reynolds number $\mathcal{R}_\lambda \longrightarrow \infty$; and third, that this dependence exhibits the scaling (2) at large $\mathcal{R}_\lambda$.

It is important to distinguish dynamically induced dependence from that due to the kinematic constraints of incompressibility and, at finite $\mathcal{R}_\lambda$, the overlap of energy and dissipation spectra. The distinction we intend is clarified by representing the total velocity field as the composition of subfields band limited in wave number k [3]. Suppose that the subfields are statistically independent of each other as is true, for example, if the total velocity field is multivariate Gaussian. There is then no statistical dependence between spatial fluctuations in bands at inertial-range k and fluctuations in bands at dissipation wave numbers. We shall show that nevertheless there is statistical dependence between δu_r and ϵ_r if the energy and dissipation spectra have some overlap.

Only the first assumption in the RSH can be examined meaningfully at the moderate $\mathcal{R}_\lambda$ available to direct numerical simulation (DNS). We use filtering operations to remove the kinematic effects due to overlap of energy and dissipation spectra and demonstrate that dynamically induced dependence, essential to RSH, does indeed survive. The low $\mathcal{R}_\lambda$ values of the DNS make impossible a meaningful test of the second and third assumptions in the RSH. Filtering operations similar to ours can be applied to real-fluid experiments at much larger $\mathcal{R}_\lambda$. This may give some valid hints about asymptotic dynamical behavior.

The RSH has never been derived from the Navier-Stokes (NS) equation, although it has been widely used in various models [4,5] of anomalous scaling of high-order velocity structure functions. Since the dissipation spectrum peaks at a wave number $k_{\text{peak}} = O(\ell_d^{-1})$, ϵ_r in (3) is dominated by dissipation-range contributions, while δu_r is an inertial-range quantity. The stretching process that carries structures of width r into structures of width ℓ_d is stochastic and not instantaneous, facts whose consequences one hopes are correctly captured by β.

Hosokawa and Yamamoto [6] were the first to test the RSH directly. They examined the correlation between δu_r and the longitudinal pseudodissipation ϵ^{11} in a DNS. Here $\epsilon^{ij} \equiv \nu(\partial u_i/\partial x_j)^2$. Experiments usually measure ϵ^{11} instead of ϵ. Praskovsky [7] and Thoroddsen and van Atta [8] found a significant correlation between δu_r and ϵ_r^{11} in wind tunnel experiments. In addition, Praskovsky showed that $\langle \delta u_r | \epsilon_r^{11} \rangle$ increases with ϵ_r^{11}. Stolovitzky, Kailasnath, and Sreenivasan [9] analyzed data taken in the atmospheric surface layer and found a strong correlation between δu_r and ϵ_r^{11}, particularly for large ϵ_r^{11}.

Chen *et al.* [10] constructed high-resolution DNS of forced stationary and decaying isotropic turbulence for $\mathcal{R}_\lambda$ between 17 and 202. They noted that δu_r and ϵ_r^{11} must be correlated even at Reynolds numbers too low to yield an inertial subrange. Wang *et al.* [11] analyzed DNS data to examine a number of matters relevant to the validity of K62: putative universal constants, scaling exponents, the probability distribution function (PDF) of β, and the use of ϵ^{11} vs ϵ.

Thoroddsen [12] suggests failure of RSH on the basis of experiments in which the statistical dependence between locally averaged dissipation rate and δu_r is much weaker if the dissipation is estimated by ϵ^{21}, the streamwise gradient of a transverse velocity component, instead of by ϵ^{11}. Stolovitzky and Sreenivasan [13] note that at least part of the statistical dependence between δu_r and ϵ_r^{11} is due to analytical constraints independent of dynamics. If the dependence effectively disappears when ϵ_r^{11} is replaced by ϵ_r^{21}, then the foundation of K62 crumbles.

In this Letter, we analyze statistical dependence between dissipation components and δu_r in DNS. Kinematic effects dominate dynamic effects for Reynolds numbers accessible to DNS. We use two devices to separate the effects. The first is to compare statistics of the DNS field with statistics of a Gaussian velocity field that has the wave number spectrum of the DNS field. The second is to employ bandpass filters in wave number to divide the DNS field into two parts with disjoint spectral supports. Cross statistics of the two parts then are free of kinematical constraints: Any realizable internal statistics of two parts are consistent with statistical independence of the two parts.

DNS of the Navier-Stokes equations on a 512^3 lattice was carried out on the CM-5 machine at Los Alamos National Laboratory. A nominal steady state was maintained by a forcing confined to wave numbers $k < 3$. The Taylor microscale $\lambda = (15\nu v_0^2/\epsilon)^{1/2}$ and microscale Reynolds number $\mathcal{R}_\lambda \equiv v_0\lambda/\nu$ were controlled by varying ν. Here v_0 is the rms value of a single vector component of velocity. Analysis was carried out for forced statistical steady states at $\mathcal{R}_\lambda = 102$, 151, and 216.

Figure 1 shows the conditional mean $\langle \delta u_r | \epsilon_r^{ij} \rangle$ as a function of ϵ_r^{ij} at separations $r = 4\pi/512$, $52\pi/512$, and $212\pi/512$ for $ij = 11$ [Fig. 1(a)] and $ij = 21$ [Fig. 1(b)]. The data shown are for $\mathcal{R}_\lambda = 216$, but flows at the other two Reynolds numbers give similar results. The chosen r values represent, respectively, a small separation close to the Kolmogorov dissipation scale l_d, a nominal inertial-range scale, and a length close to the box size. For given r, Fig. 1 illustrates that the dependence of $\langle \delta u_r | \epsilon_r^{ij} \rangle$ on ϵ_r^{ij} is strongest for $ij = 11$, confirming the observation of Thoroddsen [12]. For small r, Fig. 1(a) supports the relation $\delta u_r \sim (\epsilon_r^{11})^{1/2}$ that one obtains from Taylor expansion about $r = 0$.

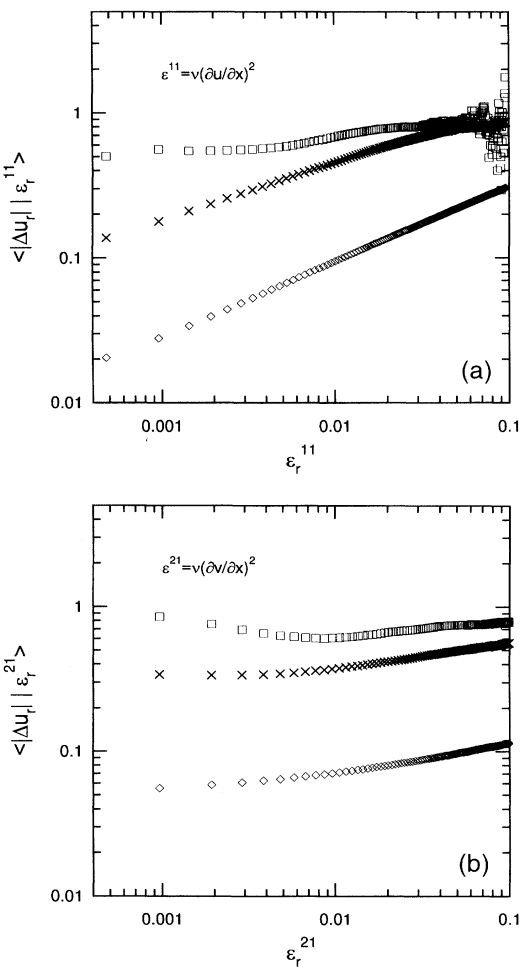

FIG. 1. $\langle \delta u_r | \epsilon_r^{ij} \rangle$ as a function of ϵ_r^{ij} for $ij = 11$ (a) and $ij = 21$ (b). The separations are $r = 4\pi/512$ ($\diamond$), $52\pi/512$ ($\times$), and $212\pi/512$ ($\square$).

The dependence of $\langle \delta u_r | \epsilon \rangle$ on the full dissipation ϵ is similar to Fig. 1(a). Figure 2 shows the correlation coefficients of the fluctuations $\delta \tilde{u}_r = \delta u_r - \langle \delta u_r \rangle$ and $\tilde{\epsilon}_r^{ij} = \epsilon_r^{ij} - \langle \epsilon_r^{ij} \rangle$. Note that ϵ^{11} yields the largest correlation coefficient. For r in the nominal inertial range, the difference in the coefficients for ϵ^{11} and ϵ^{21} or ϵ^{12} is a factor of about 2 or 3. The correlations are stronger for small separation and go to zero for very large separation.

To explore how much of the statistical dependence exhibited in Figs. 1 and 2 comes from the NS dynamics rather than kinematical constraints, we randomized phases in the DNS velocity field thereby constructing a new incompressible velocity field with the same energy spectrum, but with total statistical independence of different wave number ranges. Figure 3 shows $\langle \delta u_r | \epsilon_r^{11} \rangle$ as a function of ϵ_r^{11} for both DNS field and the random-phase field.

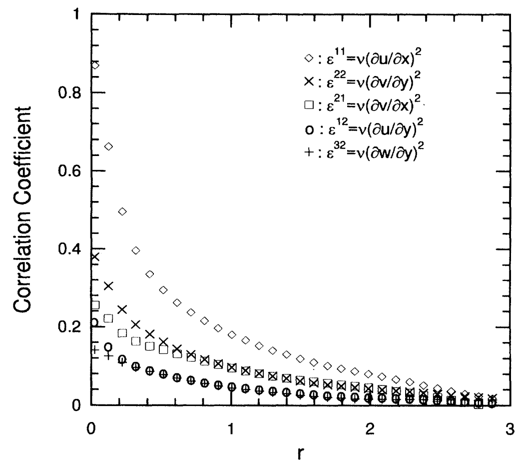

FIG. 2. Fluctuation correlation coefficients for several locally averaged dissipation components as a function of separation r.

The separations r are as in Fig. 1. The two fields show similar behavior even though their detailed spatial structures are quite different. The inset in Fig. 3 shows the correlation coefficients for these two fields. They are very close for all three r, with little effect of the phase randomization evident.

Clearly, part of the dependence predicted in the RSH is of kinematic origin in our simulations. Nevertheless, the PDF of ϵ_r is quite different for the two fields. There is a wide tail in the DNS case while the randomized field gives a narrow PDF concentrated on the mean dissipation range. The data for very large and small ϵ_r are so sparse for the random field that they are not plotted in Fig. 3.

As already mentioned, δu_r is an inertial-range quantity for inertial-range r, while the support of ϵ_r is dominated by the dissipation range. The two ranges are not fully distinct at moderate $\mathcal{R}_\lambda$. The consequent overlap of spectral support gives kinematical constraints that contribute to the statistical dependencies shown in Figs. 1–3.

What happens at very large $\mathcal{R}_\lambda$? If RSH is valid, dependence between δu_r and ϵ_r^{ij} must survive at infinite $\mathcal{R}_\lambda$. There relevant kinematic constraints are gone, and the random-phase field displays no such dependence. $\mathcal{R}_\lambda$ much higher than the values examined here cannot currently be simulated. Therefore we try to obtain some insight at modest $\mathcal{R}_\lambda$ as to whether the qualitative dynamical effects invoked by the RSH exist.

We use wave number filters to construct, from the total DNS velocity field, two fields with disjoint spectral support: $\mathbf{u}^-$ contains only wave numbers $k < k_1$ while $\mathbf{u}^+$ contains only wave numbers $k > k_2$, with $k_2 \geq k_1$. Then we construct δu_r^- and ϵ^{ij+} from the two fields. If now the DNS field is replaced by a Gaussian (random-phase) field with the same spectrum, δu_r^- and ϵ^{ij+} are statistically independent. Thus it is plausible to infer that any statistical dependence between these quantities for the actual DNS case reflects NS dynamics.

Figure 4 shows $\langle \delta \mathbf{u}_r^- | \epsilon^{ij+} \rangle$ vs ϵ^{ij+} for $ij = 11$, $ij = 21$, and $ij = 12$ at $52\pi/512$, an inertial-range separation. The DNS field is that of Fig. 1. The filter cutoff wave numbers are taken as $k_1 = k_2 = k_{\text{peak}}$. Most of the dissipation comes from $k > k_{\text{peak}}$. A statistical dependence between the $-$ and $+$ fields survives for all three choices of ij. The inset of Fig. 4 shows the correlation coefficients constructed as in Fig. 3, but now from δu_r^- and ϵ_r^{ij+}. There is zero correlation for the random-phase field ($\times$). The DNS field gives nonzero correlation for all three ij choices.

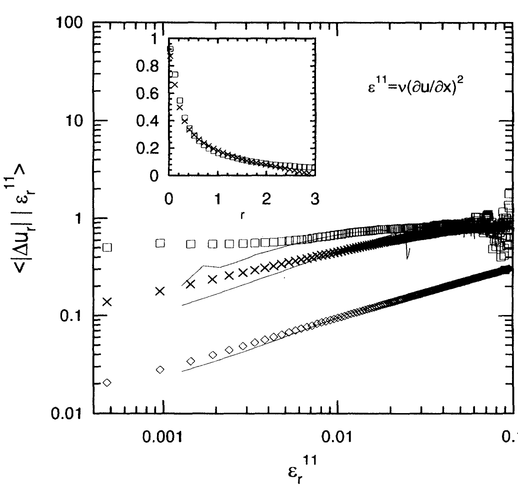

FIG. 3. $\langle \delta u_r | \epsilon_r^{11} \rangle$ as a function of ϵ_r^{11} for original velocity field (symbols) and random-phase velocity field (solid lines). The separations r and symbols are as in Fig. 1. The inset shows the correlation coefficients for original field ($\times$) and random-phase field ($\square$).

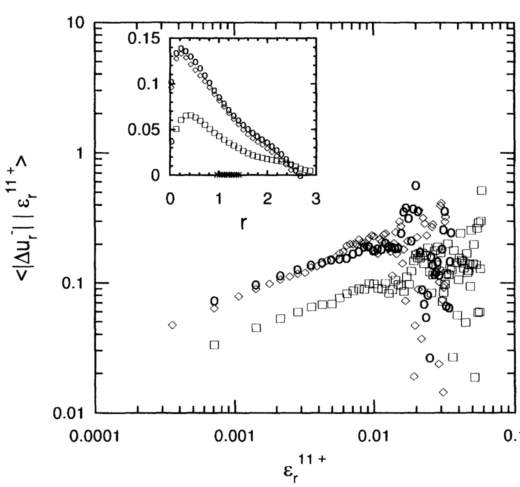

FIG. 4. $\langle \delta u_r^- | \epsilon_r^{ij+} \rangle$ vs ϵ_r^{ij+} at $r = 52\pi/512$ for $ij = 11$ ($\Diamond$), $ij = 21$ ($\bigcirc$), and $ij = 12$ ($\square$). The inset shows the corresponding correlation coefficients between dissipation-term fluctuations and velocity-difference fluctuations (the $\times$ data are for the random-phase field).

The qualitative physical picture behind RSH is that straining processes transform large-scale fluctuations in strain into dissipation-scale fluctuations. In wave number space, this corresponds to a multistep cascade process driven by low wave numbers. Therefore the physics of the RSH may be better represented by reversed conditional means like $\langle \epsilon_r^{ij+} | \Delta u_r^- \rangle$. Figure 5 shows three of these quantities vs Δu_r^- at $r = 52\pi/512$ for the same DNS data as in the preceding figures. The most striking feature is the asymmetry: For all ij, large values of ϵ_r^{ij+} are more probable for negative Δu_r^-, in accord with the well-known negativity of $\langle (\partial u/\partial x)^3 \rangle$.

The filtering operations used here can be adapted to real-life experiments at large $\mathcal{R}_\lambda$ by filtering on the longitudinal wave number instead of total wave number. The result is again two fields with disjoint spectral support, so that kinematical contributions to the RSH relations are removed.

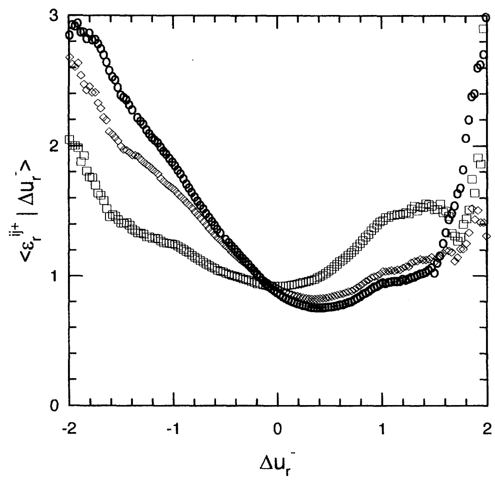

FIG. 5. $\langle \epsilon_r^{ij+} | \Delta u_r^- \rangle$ vs Δu_r^- at $r = 52\pi/512$ for $ij = 11$ ($\Diamond$), $ij = 21$ ($\bigcirc$), and $ij = 12$ ($\Box$).

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[1] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[2] A. N. Kolmogorov, C. R. Acad. Sci. **30**, 301 (1941).
[3] R. H. Kraichnan, J. Fluid Mech. **62**, 305 (1974).
[4] Z.-S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994).
[5] R. Benzi, G. Paradin, G. Parasi, and A. Vulpiadi, J. Phys. A **17**, 3521 (1984); C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987); L. C. Andrews, R. L. Phillips, B. K. Shivamoggi, J. K. Beck, and M. L. Joshi, Phys. Fluids A **1**, 999 (1989).
[6] I. Hosokawa and K. Yamamoto, Phys. Fluids A **4**, 457 (1992).
[7] A. A. Praskovsky, Phys. Fluids A **4**, 2589 (1992).
[8] S. T. Thoroddsen and C. W. van Atta, Phys. Fluids A **4**, 2592 (1992).
[9] G. Stolovitzky, P. Kailasnath, and K. R. Sreenivasan, Phys. Rev. Lett. **69**, 1178 (1992).
[10] S. Chen, G. D. Doolen, R. H. Kraichnan, and Z.-S. She, Phys. Fluids A **5**, 458 (1993).
[11] L.-P. Wang, S. Chen, J. G. Brasseur, and J. C. Wyngaard, "Examination of Hypotheses in the Kolmogorov Refined Turbulence Theory through High-Resolution Simulations" (to be published).
[12] S. T. Thoroddsen, "The Failure of the Kolmogorov Refined Similarity Hypothesis in Fluid Turbulence" (to be published).
[13] G. Stolovitzky and K. R. Sreenivasan, Rev. Mod. Phys. **66**, 229 (1994).