Reconciling the cylindrical formulation with the spherical formulation in the kinematic descriptions of collision kernel

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Kinematic descriptions of the rate of collision between two groups of particles are central to a variety of problems in cloud microphysics, engineering applications, and statistical mechanics. When particles are uniformly distributed, the collision kernel Γ depends on the statistics of relative velocities among colliding particles. In the pioneering work by Saffman and Turner ["On the collision of drops in turbulent clouds," J. Fluid Mech. **1**, 16 (1956)], two different formulations were used to calculate Γ between two arbitrary particle size groups in a turbulent flow. The first or spherical formulation is based on the radial or longitudinal component w_r of the relative velocity $\mathbf{w}$ between two particles at contact: $\Gamma^{\rm sph}=2\pi R^2\langle|w_r|\rangle$, where R is the geometric collision radius. The second or cylindrical formulation is based on the vector velocity itself: $\Gamma^{\rm cyl}=\pi R^2\langle|\mathbf{w}|\rangle$. It was shown previously by Wang *et al.* ["Statical mechanical descriptions of turbulent coagulation," Phys. Fluids **10**, 2647 (1998)] that the spherical formulation is always correct when applied to different situations, and that the cylindrical formulation overpredicts the collision kernel by about 20%–25% for collisions due to a uniform shear or due to nonuniform shears in a turbulent flow. In this paper, it is shown that the overpredictions in the cylindrical formulation are originated from the dependence of the probability distribution of $\mathbf{w}$ on the orientation of $\mathbf{R}$, and can be corrected for all situations if this orientation dependence is explicitly accounted for. A generalized cylindrical formulation is then proposed and is shown to be identical to the spherical formulation for all collision mechanisms considered in Wang *et al.* (1998). Finally, we illustrate the difference between kinematic statistics and statistics for colliding particle pairs. For example, the relative velocity for colliding particle pairs can be 30%–60% larger than the kinematic relative velocity. [DOI: 10.1063/1.1928647]

I. INTRODUCTION

Coagulational growth of small solid particles and droplets in a turbulent flow is of importance to a variety of problems in engineering and meteorology. Examples include warm-rain precipitation, cloud processing of aerosols, production of titanium-dioxide pigments, fine spray combustion, and formation of industrial emissions. The overall coagulation rate of finite-size particles in fluid turbulence is governed by three consecutive and interrelated processes: (1) geometric collision due to relative motion[1–10] and nonuniform distributions[11–15] caused by the carrier fluid turbulence, (2) collision efficiency due to local particle-particle aerodynamic or hydrodynamic interactions,[16–22] and (3) coagulation efficiency as determined by surface sticking characteristics.[23,24]

The average collision kernel Γ between two particle size groups of average number concentrations n_1 and n_2 measures the rate of collisions between the two size groups normalized by $(n_1 n_2)$. The parametrization of Γ in terms of kinematic particle-pair statistics makes it feasible to model the time evolution of particle size distribution using the Smoluchowski collision-coalescence equation (see, e.g., Pruppacher and Klett[24]). In the pioneering work by Saffman and Turner,[1] two different formulations of Γ were introduced to describe geometric collision rates between two arbitrary particle size groups in a turbulent flow. When particles are uniformly distributed, the collision kernel Γ depends on the statistics of relative velocities among colliding particles. In the first, spherical formulation, the average collision kernel is described as the average volume of fresh carrier fluid entering a collision sphere per unit time,

$$\Gamma^{\rm sph}=2\pi R^2\langle|w_r|\rangle=2\pi R^2\int|w_r|p(w_r)dw_r. \tag{1}$$

The collision sphere is defined as a sphere of radius $R=r_1+r_2$, centered on a reference particle of radius r_1. Here r_1 and r_2 are the radii of the two particle size groups, w_r is the radial or longitudinal component of $\mathbf{w}$, the velocity of an r_2 particle relative to an r_1 particle when they are separated by a distance of R. Namely, $w_r=\mathbf{w}\cdot\mathbf{R}/R$, $\mathbf{R}$ is the separation vector of magnitude equal to R. $p(w_r)$ is the probability density of w_r. One important assumption in Eq. (1) is that the relative velocity $\mathbf{w}$ is incompressible, thus influx and outflux over the surface of the collision sphere are equal. The collision kernel is then half the surface area multiplied by the average magnitude of the radial relative velocity. The validity of this assumption has been discussed for finite-inertia particles in direct numerical simulations (DNS) by Wang *et al.*[12]

In the second, cylindrical formulation, the collision ker-

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nel is described in terms of the relative velocity directly and is defined as an integrated, cylindrical volume passing through the reference particle per unit time, with cross-sectional area πR^2 and length $|\mathbf{w}|$. The average collision kernel is expressed as

$$\Gamma^{\mathrm{cyl}} = \pi R^2 \langle |\mathbf{w}| \rangle = \pi R^2 \int\int\int w\ p(\mathbf{w}) d^3\mathbf{w}, \quad (2)$$

where $w \equiv |\mathbf{w}|$, and $p(\mathbf{w})$ is the probability density of the vector velocity $\mathbf{w}$ when the pair is separated by $\mathbf{R}$. In both the above equations, the angle brackets denote averages over all orientations of $\mathbf{R}$ and spatial locations of the reference particle. We note that the cylindrical formulation, which employs the concept of a collision cylinder, is standard textbook material in statistical mechanics (see, e.g., McQuarrie[25]) and multiphase flow textbooks (e.g., Crowe *et al.*[26]).

The cylindrical formulation was viewed in Saffman and Turner[1] as an alternative rigorous formulation. The cylindrical volume due to the relative velocity $\mathbf{w}$ is often termed as *the swept volume* in atmospheric science literature and classical statistical mechanics, and the concept of swept volume has served as foundation for the physical interpretation of Γ. This cylindrical formulation is often taken as a rigorous starting point for the kinematic formulation of geometric collision kernel and has been rederived many times by others (e.g., Abrahamson,[2] Yuu[4]). Both formulations have been widely cited in the literature, and the choice between the two has been a matter of preference of the investigators.

Wang *et al.*[9] questioned whether the two formulations are equivalent for different collision mechanisms. They demonstrated that the spherical formulation is always correct when applied to different situations, and that the cylindrical formulation overpredicts the collision kernel by about 20%–25% for collisions due to a uniform shear or due to nonuniform shears in a turbulent flow. They pointed out that the two formulations are equivalent if $p(\mathbf{w})$ is everywhere the same on the surface of the collision sphere, namely, the probability density of $\mathbf{w}$ is independent of the orientation of $\mathbf{R}$; for example, collisions due to gravitational settling or due to uncorrelated relative motion such as Brownian motion. In general, Wang *et al.*[9] suggested that the cylindrical formulation should not be used for treating turbulent coagulation.

In this paper, we go one step further to demonstrate that it is possible to reconcile the cylindrical formulation with the spherical formulation if the cylindrical formulation is more carefully constructed such that the orientation dependence of $p(\mathbf{w})$ is taken into consideration explicitly. Two specific collision mechanisms for which Eqs. (1) and (2) lead to different results are considered, namely, the geometric collisions due to a uniform shear (von Smoluchowski[27]) and nonuniform shear rates in isotropic turbulence (Saffman and Turner[1]). A generalized form of the cylindrical formulation is then proposed and shown to yield the exact same result as the spherical formulation for all the collision mechanisms discussed in the work of Wang *et al.*[9] Finally, we will illustrate the difference between kinematic statistics and statistics for colliding particle pairs.

We shall limit our discussions here to the case that the particle concentrations are spatially uniform. The correction due to nonuniform concentrations in terms of the radial distribution function has been discussed in other studies (Sundaram and Collins,[11] Wang *et al.*,[12] Zhou *et al.*[13]). We also note that the correction to the kinematic formulation of Γ due to hydrodynamic interactions has been discussed recently by Wang *et al.*[22]

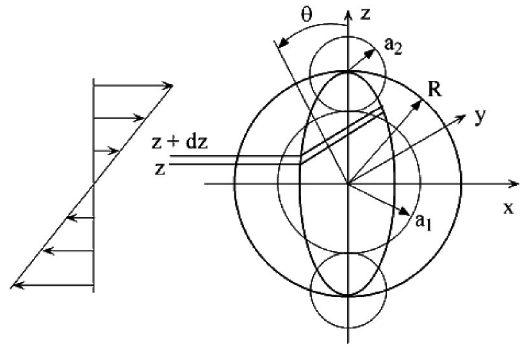

FIG. 1. Sketch for detailed analysis of collisions due to a simple shear.

II. COLLISION RATE DUE TO A UNIFORM SHEAR

The exact collision kernel of passive particles due to a uniform shear was first derived by von Smoluchowski.[27] We shall first revisit this case since it was shown in the work of Wang *et al.*[9] that the cylindrical formulation overpredicts the collision kernel even for this very simple case. The purpose here is to demonstrate how to correct the cylindrical formulation, Eq. (2), so that the exact collision kernel can be recovered.

We begin the analysis by considering the motion of size-2 particles relative to a size-1 (reference) particle. The relative velocity is given as

$$\mathbf{w} = (w_x, w_y, w_z) = (\gamma z, 0, 0) = (\gamma R \cos\theta, 0, 0), \quad (3)$$

where γ is the shear rate, z is the relative coordinate in the z direction, and θ is the polar angle (Fig. 1). While $\langle \mathbf{w} \rangle$ may be obtained easily without referring to the probability density function of $p(\mathbf{w})$, it is instructive to explicitly consider $p(\mathbf{w})$. The probability density function of $p(\mathbf{w})$ takes the form

$$p(\mathbf{w}) = p(w_x)\,\delta(w_y)\,\delta(w_z). \quad (4)$$

Therefore, Eq. (2) becomes

$$\Gamma_{12}^{\mathrm{Cyl}} = \pi R^2 \int |w_x| p(w_x) dw_x. \quad (5)$$

The probability distribution $p(w_x)$ can be obtained by the probability distribution of θ over the collision sphere. Since the area of the collision sphere is obtained in terms of θ as

$$2\pi R^2 \int_0^{\pi} \sin\theta\, d\theta,$$

it follows that the probability density of θ is

$$p(\theta) = \tfrac{1}{2}\sin\theta. \quad (6)$$

Since $dw_x = \gamma R(-\sin\theta)d\theta$, we have

$$p(w_x) = \begin{cases} 1/(2\gamma R) & \text{if } -\gamma R \leqslant w_x \leqslant \gamma R \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Therefore, the cylindrical formulation leads to

$$\Gamma_{12}^{\mathrm{Cyl}} = \pi R^2 \int_{-\gamma R}^{\gamma R} \frac{1}{2\gamma R} |w_x| dw_x = 2\pi R^2 \int_0^{\gamma R} \frac{1}{2\gamma R} w_x dw_x$$
$$= \frac{\pi}{2} \gamma R^3. \tag{8}$$

This is about 20% larger than the correct result which would have been obtained based on the spherical formulation[9]

$$\Gamma_{12}^{\mathrm{Sph}} = 2\pi R^2 \int p(w_r)|w_r| dw_r = \tfrac{4}{3}\gamma R^3. \tag{9}$$

The problem with the cylindrical formulation originates from the fact that a given w_x is only achieved at a specific θ, as such $p(w_x)$ is not independent of θ, or stated in another way, the subset of size-2 particles with a given w_x is not uniformly distributed over the collision sphere.

Now consider the actual swept volume due to a differential range of w_x. It is equal to w_x times the differential cross-section area $2\sqrt{R^2-z^2}dz = 2\sqrt{R^2-(w_x/\gamma)^2}dw_x/\gamma$. Note that the cross-section area is the projected area perpendicular to the relative motion or onto the y-z plane, referring to Fig. 1 for the geometric detail. Therefore the cylindrical formulation can be corrected in the sense that the swept volume for each w_x is summed up,

$$\Gamma_{12}^{\mathrm{Cyl,corrected}} = \int_{-\gamma R}^{\gamma R} |w_x| 2\sqrt{R^2 - (w_x/\gamma)^2} dw_x/\gamma. \tag{10}$$

This can be integrated and shown to yield the correct result as the spherical formulation, Eq. (9). The corrected formulation can be rewritten as

$$\Gamma_{12}^{\mathrm{Cyl,corrected}} = \pi R^2 \int_{-\gamma R}^{\gamma R} |w_x| \frac{1}{2\gamma R} \left[\frac{4}{\pi} \sqrt{1 - \left(\frac{w_x}{\gamma R} \right)^2} \right] dw_x. \tag{11}$$

The term in the square brackets can be viewed as a correction factor when compared to the original cylindrical formulation, Eq. (8). A plot of this correction factor is given in Fig. 2, showing that the smaller $|w_r|$ range contributes more to the collision kernel than what is assumed in the original cylindrical formulation, due to larger projected cross-sectional area. This correction is needed in addition to the consideration of $p(w_x)$. The above analysis shows that we must consider contribution by each $\mathbf{w}$ to the total swept volume carefully when $p(\mathbf{w})$ depends on the orientation of $\mathbf{R}$.

III. COLLISION RATE DUE TO NONUNIFORM SHEARS IN ISOTROPIC TURBULENCE

Next we consider collisions of small passive particles driven by nonuniform shears in an isotropic turbulence, for which the cylindrical formulation again overpredicts the geometric collision kernel.[9] Following Wang *et al.*,[9] we shall first introduce the necessary notations to describe the relative motion at the scale of R which is assumed to be smaller than

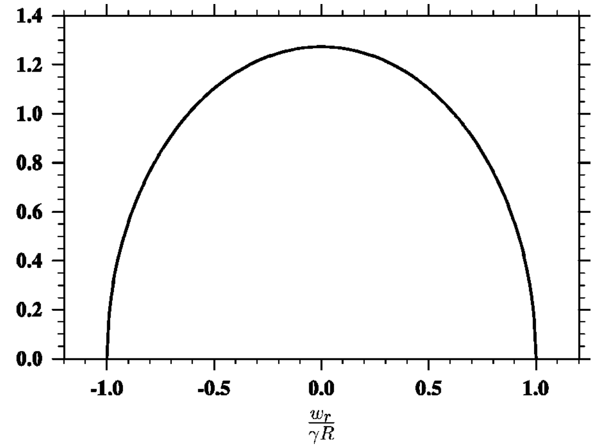

FIG. 2. The correction factor implied by Eq. (11).

the Kolmogorov scale. As in the work of Saffman and Turner,[1] the radial relative velocity is modeled as a Gaussian random variable with variance equal to

$$\langle w_r^2 \rangle = R^2 \left\langle \left(\frac{\partial u}{\partial x} \right)^2 \right\rangle = \frac{R^2}{15} \frac{\overline{\epsilon}}{\nu} = \frac{1}{15} \frac{R^2}{\tau_k^2} \equiv \sigma^2, \tag{12}$$

where u is the x component of the fluid velocity, τ_k is the Kolmogorov time scale, ν is the fluid kinematic viscosity, and $\overline{\epsilon}$ is the average rate of viscous dissipation per unit mass. In reality the probability distribution of relative velocity may deviate significantly from the Gaussian distribution.[28,29] It is assumed here so that the collision kernel can be explicitly obtained in closed forms and as such different formulations can be compared. The major conclusions of the paper are not affected by this assumption.

In terms of velocity components in a spherical coordinate system over the collision sphere, the probability distribution of $\mathbf{w}$ can be expressed as[9,28]

$$p(\mathbf{w}) = \frac{1}{2(\sqrt{2\pi}\sigma)^3} \exp\left(-\frac{w_r^2}{2\sigma^2} - \frac{w_\theta^2 + w_\phi^2}{4\sigma^2} \right)$$
$$= \frac{1}{2(\sqrt{2\pi}\sigma)^3} \exp\left(-\frac{w_r^2}{4\sigma^2} - \frac{w^2}{4\sigma^2} \right). \tag{13}$$

The dependence of $p(\mathbf{w})$ on the orientation of $\mathbf{R}$ is evident since

$$w_r = w_z \cos\theta + (w_x \cos\phi + w_y \sin\phi) \sin\theta,$$

where θ is the polar angle and ϕ is the azimuthal angle. The dependence is originated from the fact that the statistics of longitudinal velocity derivative differ from the statistics of transverse velocity derivative.[9,28] Previously, it was shown that this dependence causes the cylindrical formulation to overpredict the collision kernel by about 25%.[9]

Here we shall demonstrate that it is actually possible to reconcile the cylindrical formulation with the spherical formulation. To proceed, we consider the contribution of a given $\mathbf{w}$ to the collision kernel. We shall redefine the polar angle relative to the direction of $-\mathbf{w}$ (see Fig. 3). Then $w_r = -w \cos\theta$ and Eq. (13) can be written as

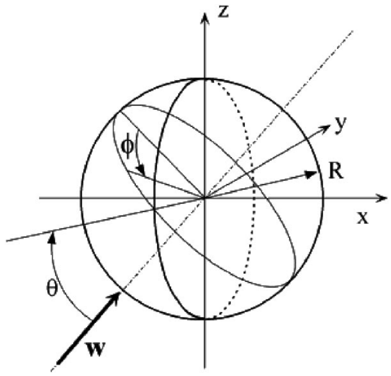

FIG. 3. Notation for the generalized cylindrical formulation.

$$p(\mathbf{w};\theta) = \frac{1}{(2\pi)^{1.5}2\sigma^3}\exp\left[-\frac{w^2(1+\cos^2\theta)}{4\sigma^2}\right]. \quad (14)$$

Therefore, for a given subset of size-2 particles with relative velocity $\mathbf{w}$, the explicit dependence of $p(\mathbf{w})$ on the orientation of $\mathbf{R}$, as measured by θ, has been made explicit by the above expression; namely, the variance of $\mathbf{w}$ varies with θ. For this reason, the swept volume must be computed carefully for each value of θ. Consider the differential region on the collision sphere with the polar angle changing from θ to $\theta+d\theta$. The area of this differential surface is $dA = (2\pi R\sin\theta)(R\,d\theta) = 2\pi R^2\sin\theta\,d\theta$. The net swept volume through this differential area is equal to w times $dA\cos\theta$ (i.e., the projected area in the plane perpendicular to $\mathbf{w}$). Therefore, in terms of the swept volume, we can generalize Eq. (2) as

$$\Gamma^{\text{Cyl,corrected}} = \int\int\int w\, d^3\mathbf{w}\int_0^{\pi/2} d\theta\, p(\mathbf{w};\theta)2\pi R^2 \times\sin\theta\cos\theta, \quad (15)$$

where $p(\mathbf{w},\theta)$ is the probability density of $\mathbf{w}$ at the specific orientation angle θ. Note that only half of the collisional sphere with inward flux as defined by $0\leqslant\theta\leqslant\pi/2$ needs to be considered. The above equation would reduce exactly to the original cylindrical formulation, Eq. (2), if $p(\mathbf{w};\theta)$ were assumed to be independent of θ.

Substituting Eq. (14) into Eq. (15), we can carry out the integration over θ, yielding

$$\Gamma^{\text{Cyl,corrected}} = \frac{1}{\sqrt{2\pi}}\frac{R^2}{\sigma}\int\int\int\frac{1}{w}\left[\exp\left(-\frac{w^2}{4\sigma^2}\right) - \exp\left(-\frac{w^2}{2\sigma^2}\right)\right]d^3\mathbf{w}. \quad (16)$$

Since the integrand in Eq. (16) only depends on the magnitude of $\mathbf{w}$, we can write $d^3\mathbf{w}=4\pi w^2dw=2\pi w dw^2$ and the above expression becomes

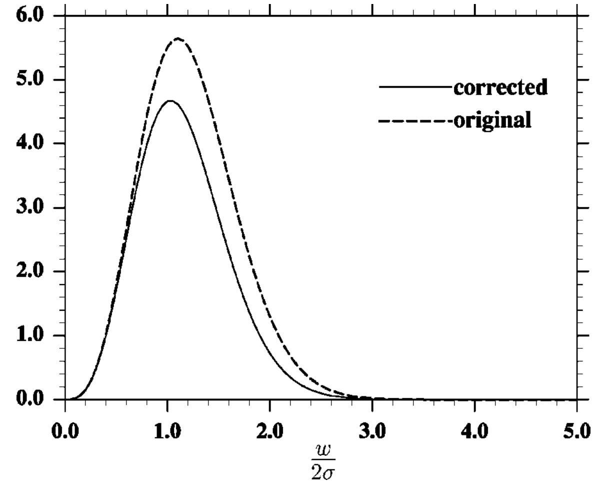

FIG. 4. Comparison of the two integrands in Eqs. (20) and (22).

$$\Gamma^{\text{Cyl,corrected}} = \sqrt{2\pi}\frac{R^2}{\sigma}\int_0^\infty\left[\exp\left(-\frac{w^2}{4\sigma^2}\right) - \exp\left(-\frac{w^2}{2\sigma^2}\right)\right]dw^2, \quad (17)$$

which can be integrated to obtain

$$\Gamma^{\text{Cyl,corrected}} = 2\sqrt{2\pi}R^2\sigma = \sqrt{\frac{8\pi}{15}}\frac{R^3}{\tau_k}. \quad (18)$$

Therefore, the correct result based on the spherical formulation[1,9] is recovered.

To quantify the difference between the original cylindrical formulation and the corrected cylindrical formulation, we can rewrite Eq. (17) as

$$\Gamma^{\text{Cyl,corrected}} = R^2\sigma\int_0^\infty F^{\text{Cyl,corrected}}\left(\frac{w}{2\sigma}\right)\frac{dw}{2\sigma}, \quad (19)$$

where the integrand is

$$F^{\text{Cyl,corrected}}(x) = 8\sqrt{2\pi}x\left[e^{-x^2} - e^{-2x^2}\right]. \quad (20)$$

While the original formulation can be written similarly but with a different integrand, according to the Appendix in the work of Wang *et al.*,[9] as

$$\Gamma^{\text{Cyl}} = R^2\sigma\int_0^\infty F^{\text{Cyl}}\left(\frac{w}{2\sigma}\right)\frac{dw}{2\sigma}, \quad (21)$$

$$F^{\text{Cyl}}(x) = 4\pi\sqrt{2}x^2\text{erf}(x)e^{-x^2}, \quad (22)$$

where the standard error function is defined as $\text{erf}(x) = (2/\sqrt{\pi})\int_0^x\exp(-u^2)du$. In Fig. 4, the two integrands are compared, showing that the original formulation overpredicts the contributions from intermediate to large magnitudes of relative velocities. The net effect is a roughly 25% overprediction of the collision kernel.[9]

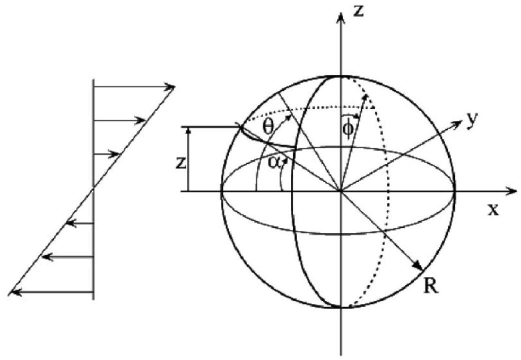

FIG. 5. Notation for the generalized formulation when applied to collisions due to a simple shear.

IV. THE GENERALIZED CYLINDRICAL FORMULATION

In light of the above discussions and derivations, we shall now propose a generalized cylindrical formulation as follows:

$$\Gamma^{\text{Cyl,generalized}} = \int\int\int d^3\mathbf{w} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, p(\mathbf{w};\theta,\phi) \times (R^2 \sin\theta\cos\theta) w, \tag{23}$$

where $p(\mathbf{w};\theta,\phi)$ denotes the probability density of $\mathbf{w}$ at a specific orientation on the surface of the collision sphere as defined by a polar angle θ, relative to the direction of $-\mathbf{w}$, and an azimuthal angle ϕ (see Fig. 3). By definition,

$$\int\int\int d^3\mathbf{w}\; p(\mathbf{w};\theta,\phi) = 1,$$

for any θ and ϕ combination. The integrand in Eq. (23) can be viewed as the product of three groups: (1) the differential cross-sectional area, perpendicular to $\mathbf{w}$, given by $(R\,d\theta) \times (R\sin\theta\,d\phi)(\cos\theta)$; (2) the relative swept distance per unit time given by w; (3) the percentage of the size-2 particles seen by a size-1 particle with relative velocity ranging from $\mathbf{w}$ to $\mathbf{w}+d^3\mathbf{w}$, given by $p(\mathbf{w};\theta,\phi)d^3\mathbf{w}$. Once again, for a given $\mathbf{w}$, only half of the collision sphere surface where inward fluxes are realized, as defined by $0 \le \theta \le \pi/2$, needs to be considered (Fig. 3).

Equation (23) is simply a generalization of Eq. (15). When applied to isotropic turbulence discussed in the last section, $p(\mathbf{w};\theta,\phi)$ is independent of ϕ, Eq. (23) reduces to Eq. (15).

We shall now demonstrate that Eq. (23) also applies to the case of collisions due to a simple uniform shear which was discussed in Sec. II. Referring to Fig. 5, for a given w_x, we can write

$$p(\mathbf{w};\theta,\phi) = \delta(w_x - \gamma R\cos\alpha)\,\delta(w_y)\,\delta(w_z), \tag{24}$$

where $\cos\alpha \equiv w_x/(\gamma R)$. With the definition of θ and ϕ as shown in Fig. 5, we have $R\cos\alpha = R\sin\theta\cos\phi$. Therefore, Eq. (23) becomes

$$\Gamma^{\text{Cyl,generalized}} = 2\int_0^{\gamma R} w_x dw_x \int_{-\alpha}^{\alpha} d\phi \int_{\alpha}^{\pi/2} d\theta \times \delta(w_x - \gamma R\sin\theta\cos\phi)(R^2\sin\theta\cos\theta). \tag{25}$$

In writing the above, we have recognized the fact that the contribution to the collision kernel from $-\gamma R \le w_x \le 0$ is the same as the contribution from $0 \le w_x \le \gamma R$. The above integral can be carried out analytically (see the Appendix) to yield

$$\Gamma^{\text{Cyl,generalized}} = \tfrac{4}{3}\gamma R^3, \tag{26}$$

which recovers the expected result of the collision kernel due to a uniform shear.

It is trivial to show that the generalized form reduces to the original cylindrical formulation if $p(\mathbf{w};\theta,\phi)$ is independent of θ and ϕ. We can cite two examples here. The first is the collision kernel due to gravitational settling, for which $p(\mathbf{w};\theta,\phi)$ is given by

$$p(\mathbf{w};\theta,\phi) = \delta(w_x)\,\delta(w_y)\,\delta(w_z - \Delta V), \tag{27}$$

where ΔV is the differential settling rate.

The second example is collisions due to Brownian motion or between random molecules. In this case, we have

$$\langle w_r^2\rangle = \langle w_\theta^2\rangle = \langle w_\phi^2\rangle \equiv \sigma^2 \tag{28}$$

and

$$p(\mathbf{w};\theta,\phi) = \frac{1}{(\sqrt{2\pi}\sigma)^3}\exp\left(-\frac{w^2}{2\sigma^2}\right). \tag{29}$$

Therefore, for both the gravitational collision and Brownian collision, $p(\mathbf{w};\theta,\phi)$ is independent of the orientation of $\mathbf{R}$. As shown previously,[9] the original cylindrical formulation and the spherical formulation then give identical results.

Finally, we shall give the generalized cylindrical formulation, Eq. (23), an interpretation that would relate it directly to the spherical formulation given by Eq. (1). Rewriting Eq. (23) as

$$\Gamma^{\text{Cyl,generalized}} = (2\pi R^2)\int\int\int d^3\mathbf{w}\frac{1}{2\pi R^2}\int_0^{2\pi} d\phi \times \int_0^{\pi/2} d\theta (R^2\sin\theta) p(\mathbf{w};\theta,\phi)|w_r|, \tag{30}$$

where $|w_r| \equiv w\cos\theta$ denotes the magnitude of inward radial relative velocity corresponding to a particular combination of $\mathbf{w}$, θ, and ϕ. The combination $[(2\pi R^2)^{-1} d\phi\, d\theta (R^2 \sin\theta)]$ represents the probability of finding a pair at a specific location of the collision sphere if the pair distribution density is independent of the orientation of $\mathbf{R}$. It is then evident that the combined integration provides a means to compute the average value of w_r over all realizations of the contacting pairs, as averages over all possible $\mathbf{w}$ and all possible orientations for a given $\mathbf{w}$ have been considered. This shows that Eq. (23) is essentially equivalent to Eq. (1), but with an explicit expression on how the average of w_r should be obtained.

V. THE AVERAGE RELATIVE VELOCITY OF COLLIDING PARTICLE PAIRS

Here we shall comment on the average radial relative velocity $\langle |w_r|_c \rangle$ based on only those particle pairs which participate in collision events. This radial relative velocity is referred to as the normal collision velocity in the study of Mei and Hu[8] or the radial relative velocity for colliding pairs in the work of Wang *et al.*[12] For an isotropic Gaussian turbulence, the analysis of Mei and Hu[8] showed that $\langle |w_r|_c \rangle = 1.58 \langle |w_r| \rangle$, while the theory of Wang *et al.*[12] led to $\langle |w_r|_c \rangle = 1.57 \langle |w_r| \rangle$. Here $\langle |w_r| \rangle$ is the kinematic average radial relative velocity used in the collision kernel formulation, and is computed using all pairs that are separated at the near at-contact distance (see, e.g., the work of Wang *et al.*[12]). In the spherical formulation, the two different relative velocity statistics are given as[8,12]

$$\langle |w_r| \rangle = \int w_r p(w_r) dw_r, \tag{31}$$

$$\langle |w_r|_c \rangle = \frac{\int w_r^2 p(w_r) dw_r}{\int w_r p(w_r) dw_r}. \tag{32}$$

We note that only the kinematic relative velocity $\langle |w_r| \rangle$ is needed for the kinematic formulation of the collision kernel. However, the radial relative velocity for colliding pairs, $\langle |w_r|_c \rangle$, must be used when addressing the consequence of particle-particle collisional interactions, such as local hydrodynamic interactions and collision and coalescence efficiencies (see, e.g., the work of Wang *et al.*[22]). These near-field interactions determine whether two approaching particles can overcome the additional viscous resistance to take advantage of the molecular attractive force to actually result in a coalescence or simply bounce off (see, e.g., the work of Hocking[30] and Jonas[31]). The relative approaching velocity or average kinetic energy of geometrically colliding particles plays a key role as this sets the initial conditions for the analysis of near-field interactions. As emphasized in the work of Mei and Hu[8] and Wang *et al.*,[12] the two relative velocities are quite different in magnitude and they should be acquired separately.[8,12]

In the spirit of the generalized cylindrical formulation, Eq. (30), we can also write

$$\langle |w_r| \rangle = \frac{1}{2\pi} \int \int \int d^3\mathbf{w} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \times p(\mathbf{w};\theta,\phi) w \cos\theta, \tag{33}$$

and

$$\langle |w_r|_c \rangle = \frac{\int \int \int d^3\mathbf{w} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \, p(\mathbf{w};\theta,\phi) w^2 \cos^2\theta}{\int \int \int d^3\mathbf{w} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \, p(\mathbf{w};\theta,\phi) w \cos\theta}. \tag{34}$$

We can show that Eq. (33) is identical to Eq. (31), and that Eq. (34) is identical to Eq. (32).

For the case of simple shear flow discussed in Sec. II, we can obtain, using either the spherical formulation or the generalized cylindrical formulation, that

$$\langle |w_r| \rangle = \frac{2}{3\pi} \gamma R \tag{35}$$

and

$$\langle |w_r|_c \rangle = \frac{\pi}{10} \gamma R = \frac{3\pi^2}{20} \langle |w_r| \rangle = 1.480 \langle |w_r| \rangle. \tag{36}$$

Therefore, $\langle |w_r|_c \rangle$ is 48% larger than $\langle |w_r| \rangle$.

For isotropic Gaussian turbulence, we can obtain

$$\langle |w_r| \rangle = \sqrt{\frac{2}{\pi}} \sigma \tag{37}$$

and

$$\langle |w_r|_c \rangle = \sqrt{\frac{\pi}{2}} \sigma = \frac{\pi}{2} \langle |w_r| \rangle = 1.5708 \langle |w_r| \rangle. \tag{38}$$

Obviously, the original, incorrect cylindrical formulation will be unable to produce the correct statistics for $\langle |w_r| \rangle$ and $\langle |w_r|_c \rangle$, as expected from the extensive discussions on the collision kernel given in previous sections. Overpredictions of the collision kernel by the original cylindrical formulation lead to exactly the same levels of overpredictions of $\langle |w_r| \rangle$, as the collision kernel and the average relative velocity are proportional to one another.

VI. CONCLUSIONS

In this short paper, we demonstrate that the dependence of the probability distribution of $\mathbf{w}$ on the orientation of $\mathbf{R}$ exists for particle collisions due to a simple uniform shear or nonuniform shears in an isotropic turbulence. For these cases, the concept of the swept volume can still be applied, but careful considerations of the orientational dependence are necessary to correctly express the amount of the swept volume. In this manner, the cylindrical formulation can be corrected to give identical results as the spherical formulation. The analyses in this paper also emphasize the importance of relating the swept volume (a concept in the cylindrical formulation) to differential regions over the collision sphere (a concept in the spherical formulation). Furthermore, a generalized cylindrical formulation is proposed and is shown to be identical to the spherical formulation for all collision mechanisms considered by Wang *et al.*[9] In other words, orientation-dependent probability density of $\mathbf{w}, p(\mathbf{w};\theta,\phi)$, is needed to correct the cylindrical formulation.

We hope that the detailed analyses provided in this short paper further clarify the origin of errors in the commonly used cylindrical formulation, although the level of errors is perhaps tolerable for most engineering and meteorological applications.

The two collision mechanisms analyzed in detail here were only applied to passive (inertialess) particles, it is expected that the concept and the generalized formulation should be equally applicable to inertial particles when the preferential concentration effect is taken into account in terms of the radial distribution function.[11–15] A generalization of the radial distribution function in terms of the orientation angles of $\mathbf{R}$ would be necessary in general.

Finally, a distinction between kinematic statistics and statistics for colliding particle pairs was also made. For example, the radial relative velocity for colliding particle pairs can be 30%–60% higher than the kinematic radial relative velocity.

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APPENDIX: DERIVATION OF EQ. (26) FROM EQ. (25)

Introducing $s=\sin\theta$ and using the property of the delta function, we can carry out the integration over θ as

$$\int_{\alpha}^{\pi/2} d\theta\, \delta(w_x - \gamma R \sin\theta\cos\phi)(R^2 \sin\theta\cos\theta)$$
$$= \int_{\sin\alpha}^{1} ds\, \delta(w_x - \gamma R s\cos\phi)(R^2 s)$$
$$= \frac{1}{(\gamma\cos\phi)^2}\int_{\sin\alpha}^{1} (\gamma R s\cos\phi)$$
$$\times\delta(w_x - \gamma R s\cos\phi) d(\gamma R s\cos\phi) = \frac{1}{(\gamma\cos\phi)^2} w_x. \tag{A1}$$

Therefore, Eq. (21) can be written as

$$\Gamma^{\text{Cyl,generalized}} = 2\gamma R^3 \int_0^1 \rho^2 d\rho \int_{-\alpha}^{\alpha} \frac{1}{\cos^2\phi} d\phi$$
$$= 2\gamma R^3 \int_0^1 \rho^2 d\rho\; 2\int_{\rho}^{1} \frac{dq}{q^2\sqrt{1-q^2}},$$

with dummy variables $\rho \equiv w_x/(\gamma R)$ and $q \equiv \cos\phi$. Exchanging the order of integration, we can carry out the integration as follows:

$$\Gamma^{\text{Cyl,generalized}} = 4\gamma R^3 \int_0^1 \frac{dq}{q^2\sqrt{1-q^2}} \frac{q^3}{3} = \tfrac{4}{3}\gamma R^3. \tag{A2}$$

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