

Energy spectrum in the dissipation range of fluid turbulence

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High-resolution, direct numerical simulations of three-dimensional incompressible Navier–Stokes equations are carried out to study the energy spectrum in the dissipation range. An energy spectrum of the form $A(k/k_d)^\alpha \exp[-\beta k/k_d]$ is confirmed. The possible values of the parameters α and β , as well as their dependence on Reynolds numbers and length scales, are investigated, showing good agreement with recent theoretical predictions. A ‘bottleneck’-type effect is reported at $k/k_d \approx 4$, exhibiting a possible transition from near-dissipation to far-dissipation.

1. Introduction

The study of the statistical properties of fluid turbulence at small scales is very difficult from both experimental and numerical points of view. In the former, the signal at these scales is exceedingly weak and consequently is easily contaminated by the background noise that is naturally present in experiments. Resolution of inertial-range and dissipation-range scales at high Reynolds numbers by direct numerical simulations (DNS) requires computational capabilities far beyond those available today.

In the last four decades, several functional forms for the energy spectrum in the dissipation and far-dissipation ranges have been proposed based on heuristic arguments. The energy spectrum in isotropic turbulence can be written as $E(k) = \epsilon^{2/3} k^{-5/3} F(k/k_d)$, consistent with Kolmogorov’s (1941) ideas. In the expression above, $E(k) = 4\pi k^2 \langle \frac{1}{2} |\hat{u}(k)|^2 \rangle$, where $\hat{u}(k)$ is the Fourier transform of the velocity field, $k_d = \epsilon^{1/4} \nu^{-3/4}$ is the Kolmogorov dissipation wave number, which represents the scale at which the relative importance of the inertial and dissipative terms is comparable, ϵ is the mean rate of energy dissipation per unit mass, and ν is the kinematic viscosity. $\langle \rangle$ denotes an ensemble average. For scales associated with wavenumbers much smaller than k_d , the energy spectrum becomes essentially independent of the viscosity ν , and, in that limit, $F(k/k_d)$ is the Kolmogorov constant C_K . For wavenumbers $k > k_d$,

dissipation is important and the function F has a strong dependence on the viscosity through k_d .

A widely proposed form for $E(k)$ in the dissipation range is

$$E(k) \sim (k/k_d)^\sigma \exp[-\beta(k/k_d)^n].$$

There has been considerable debate as to what the values of the parameters α , β and n are. Townsend (1951) and Novikov (1961) suggested $n = 2$, based on a Taylor expansion of the velocity field at scales much smaller than $\eta \approx 1/k_d$, the dissipation length scale. This same value is preferred by Smith and Reynolds (1991) in more recent work in which they compare the skewness for the longitudinal velocity derivative obtained from experiments with the values provided by $E(k) \sim (k/k_d)^\alpha \exp[-\beta(k/k_d)^n]$, with $n = 1$, $\frac{4}{3}$ and 2, and $\alpha = -\frac{5}{3}$. They conclude that $n = 2$ is the best fit, and that $n = 1$ is a poor model. Sanada (1992) proposed a spectrum with $n = 1$, but with a sharp change at a transitional wavenumber. His model fitted the spectrum from simulations, and yielded a skewness of the velocity derivative in agreement with experimental values. Manley (1992) argued that the measurements used by Smith and Reynolds did not extend much beyond the Kolmogorov scale η , and accordingly he introduced a cutoff wavenumber for the computation of the skewness. With this correction, $n = 1$ is a better fit for the experimental data than $n = 2$. Kraichnan's (1959) direct interaction approximation (DIA) predicts $n = 1$ for the far-dissipation range; Orszag (1977) argued for $n \leq 1$, and Foias *et al.* (1990) concluded that $n \geq 1$ from their analysis. Sreenivasan (1985) found experimentally $n = 1$ (for $0.5 < k/k_d < 1.5$), and Chen *et al.* (1993) observed that the same value would best fit the energy spectrum, in a highly resolved DNS at a relatively low Reynolds number. Thus the value of $n = 1$ seems to be established from theories, experiments and numerical simulations. In the rest of this paper we shall concentrate on an energy spectrum of the form

$$E(k) \sim (k/k_d)^\alpha \exp[-\beta(k/k_d)]. \quad (1)$$

There are only few studies for determining α and β . Some phenomenological models (McComb 1990) suggest $\alpha = -\frac{5}{3}$, the value that has the attractive property of being consistent with Kolmogorov's scaling for lower wavenumbers. On the other hand, both DIA (Kraichnan 1959) and the quasi-normal theories (Orszag 1977) predict $\alpha = 3$. Recent numerical studies by Kida *et al.* (1992) with data for $k/k_d < 4$ found that $\alpha < 0$. Kida *et al.* argue that the negative value for α is connected to the fact that in the dissipation range, the energy transfer is dominated by nonlocal triads. See also the study by Zhou (1994). Domaradzki (1992) obtains $\alpha = -2$, deduced from a scaling ansatz for energy transfer. On the other hand, Chen *et al.* (1993) have confirmed the prediction of the DIA and the EDQNM for $4 < k/k_d < 10$, obtaining $\alpha = 3.3$ and $\beta = 7.1$. Since the Reynolds numbers have been so low, some concerns still remain. Using very large wind tunnel experiments, Saddoughi and Veeravalli (1994) found $\beta = 5.2$, for $0.2 < k/k_d < 1$, at $R_\lambda \approx 600$, in accordance with other work (see e.g. Kida and Murakami 1987, Sanada 1992; Kida *et al.* 1992).

In this paper, we concentrate on the determination of α and β , and their possible dependence on the Reynolds number and wavelength, from numerical simulations. A series of direct numerical simulations of incompressible Navier–Stokes equations has been carried out in a periodic box with sides 2π .

Table 1. Direct numerical simulations on a $(2\pi)^3$ periodic domain. Resolution ranges are 128^3 , 256^3 and 512^3 . The third entry corresponds to a 512^3 resolution, double-precision run

Run	R_λ	k_d	k_{\max}/k_d	Resolution
1	20.9	9.7	10	256
2	33.0	13.9	8	256
3	38.0	16.6	12	512
4	39.1	16.7	7	256
5	48.4	19.9	5.8	256
6	52.9	23.6	4.7	256
7	67.9	33.4	3.3	256
8	101.9	55.7	4.4	512

We used a pseudo-spectral code on the CM-5 parallel supercomputer at Los Alamos National Laboratory. The resolution ranges from 256^3 (maximum wavenumber $k_{\max} = 128$) to 512^3 ($k_{\max} = 256$). The system is initialized with a random velocity field with Gaussian statistics. These Fourier velocity amplitudes give an energy spectrum of $k^4 \exp[-k/k_0]^2$, with $k_0 \approx 5$. If the system is then left to evolve, nonlinear interactions will redistribute the energy among all available modes. This familiar process delivers energy (in a statistical sense) from larger scales to smaller scales, until the excited wavenumbers are high enough for the viscosity to efficiently dissipate energy. In order to achieve a steady state, energy is pumped into the system by stirring it at the large scales. Specifically, the mode amplitudes for $k = 1$ and 2 (the largest scales) are forced to values consistent with Kolmogorov's $-\frac{5}{3}$ law for the energy spectrum. Once statistically steady values of R_λ and dissipation per unit mass, ϵ , were reached, several snapshots of the energy spectrum, spanning several eddy-turnover characteristic times, were averaged and analyzed.

In Table 1, the simulations are sorted by increasing $R_\lambda = v_0 \lambda / \nu$, the Taylor microscale Reynolds number, where v_0 is the mean square velocity and λ is the Taylor microscale. The table also displays information about how well the dissipation range is resolved (column k_{\max}/k_d). Notice that although for some of the simulations k_{\max} is of the order of k_d , for six of the simulations $k/k_d \gtrsim 5$, i.e. fairly deep into the dissipation range. In particular, the third row in the table ($R_\lambda = 38$) corresponds to a double-precision, 512^3 DNS, which, to the best knowledge of the authors, constitutes the memory-wise most demanding turbulence simulation to date.

To investigate the possible values for α and β , the quantity $d \ln E(k) / d \ln k$, as a function of k/k_d , was computed from the energy spectrum for each of the runs. If the functional form suggested in (1) is correct then $d \ln E(k) / d \ln k$ should be close to a straight line. The slope β/k_d and the intersection of the line with the vertical axis (at $d \ln E(k) / d \ln k = \alpha$) were then determined by means of a least-squares fit to restricted ranges of k/k_d .

Figure 1 shows $d \ln E(k) / d \ln k$ versus k/k_d for the run that best resolves the dissipation range (run 3 in Table 1). Two different slopes are found for the lower- and higher-wavenumber regions, the latter being steeper. The transition occurs at $k/k_d \approx 4$. These are common features for the simulations that penetrate deep into the dissipation range, namely the two-slope behaviour for

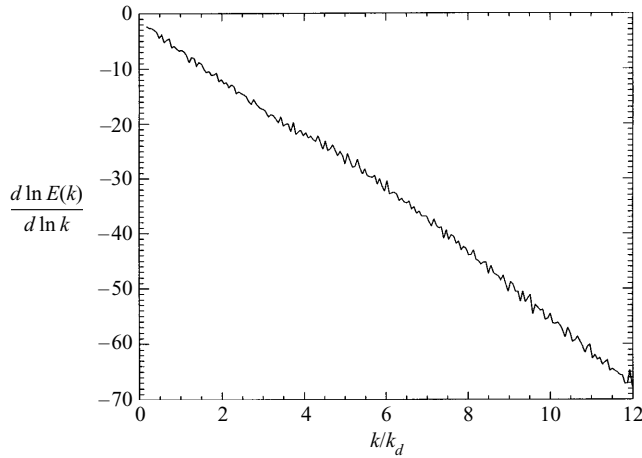


Figure 1. $d \ln E / d \ln k$ versus k/k_d for run 3 in Table 1. Two slopes are observed in the low- and high-wavenumber ranges. The energy spectrum $E(k)$ is computed by averaging the energy spectrum over several eddy-turnover times after the system has reached a steady state.

$d \ln E(k) / d \ln k$ versus k/k_d , and the transition between the two slopes at $k/k_d \approx 4$. It is possible that this shift in slope indicates a change in the nature of the balance of the triadic interactions. We have calculated the transfer energy function $T(k|p, q)$ that measures the amount of energy transferred to wavenumber k due to interactions with wavenumbers p and q . A preliminary study of $T(k|p, q)$ has not proved to be very revealing.

A bump at $k/k_d \approx 4$ in Fig. 1 accompanies the occurrence of a change in the energy slope. Similar bumps at much smaller k/k_d have been ascribed to a ‘bottleneck’ effect (Lohse and Müller-Groeling 1995) in which triadic interactions that can cascade energy out of the region are inhibited because the spectrum falls off rapidly as k/k_d increases. The energy pile-up in our study seems to be of a weaker nature than that reported by Lohse and Müller-Groeling. In that work, the second-order derivative of the energy with respect to k is non-monotonic. In our study, the third derivative is the non-monotonic one. A completely satisfying explanation of this effect is more difficult, since there is no clearly conserved quantity associated with the dissipation range, whereas, in the inertial range, the energy flux is independent of the wavenumber, i.e. it is conserved. It could be conjectured through these observations that the sharp change in the local slope of the energy spectrum will result in an energy pile-up close to the transition point, due to the change of the nature of the balance of the triadic interactions.

From the energy spectrum shown in Fig. 1, we computed α and β as functions of k/k_d . The results are displayed in Fig. 2. Panels (a) and (b) show α and β respectively as functions of k/k_d for run 3 in Table 1. These values were obtained as the local α and β (i.e. local intersection with the y axis and local slope) of $d \ln E / d \ln k$. The values for α and β assigned to a particular wavenumber were computed as least-squares fits to the energy spectrum in the range $[k/k_d - 0.9, k/k_d + 0.9]$. For α , we observe a tendency to be negative for lower k and positive for higher wavenumbers. The plot in (b) shows that β is

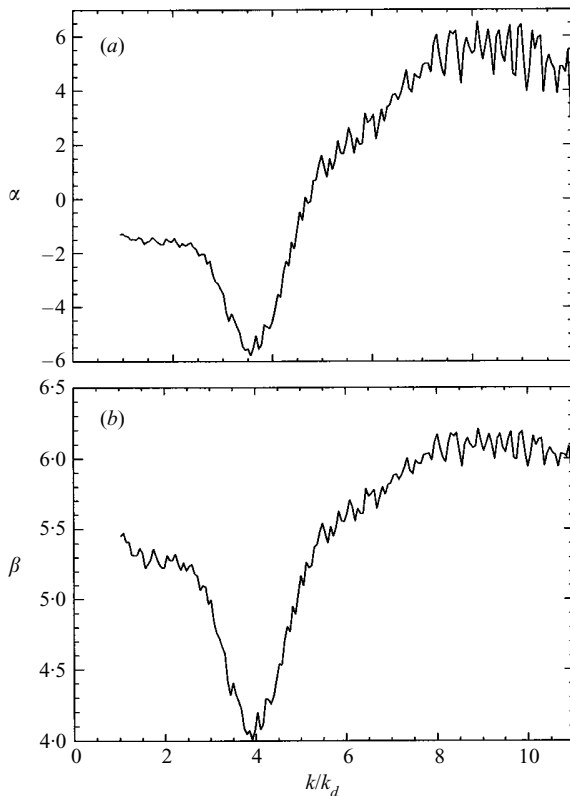


Figure 2. Local values of α and β as functions of k/k_d , calculated for run 3 in Table 1.

fairly constant at a value between five and six, except at $k/k_d \approx 4$, where it drops to about four. For both parameters, we observe that a plateau is reached for $k/k_d < 2$ and $k/k_d > 8$. We note that this plateau is not expected to extend to the inertial range scales, since that would imply that the energy spectrum would have the same functional form in the dissipation and inertial ranges.

Figure 3 shows the dependence of the coefficients α and β on R_λ . Panel (a) displays α versus R_λ . Points indicated by crosses correspond to measurements done for $k/k_d < 3$ for all runs in Table 1; diamonds correspond to $k/k_d > 6$ for runs 1 to 4. For α , all values obtained from the range $k/k_d < 3$ are consistently negative, within the range -1 to -2 , in the neighbourhood of the Kolmogorov value $-\frac{5}{3}$. The four points obtained in the more dissipative range are positive, in the range 3–6. This two-valued behaviour of α is a preliminary result. It might be possible that the region of positive values for α can be extended beyond $R_\lambda \approx 40$ –50 if these values are evaluated from computations that resolve the dissipation range better. In other words, it is possible that the negative values for α appear because they are being computed from the near-dissipation range.

For β , the values measured in the range $k/k_d > 6$ (diamonds), are consistently higher than those measured in the range $k/k_d < 3$. Recently, Sirovich *et al.* (1994) derived the energy spectrum (for both inertial range and dissipation range) from the third-order structure function. The value of β obtained in that

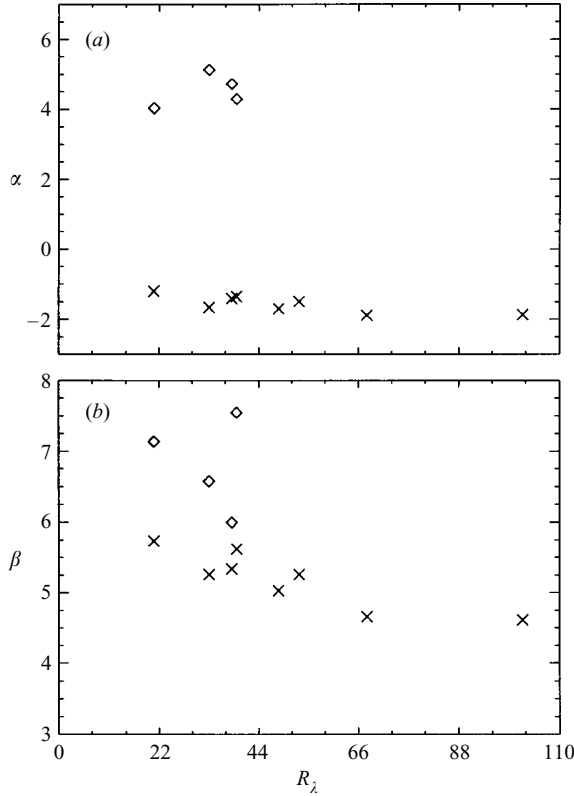


Figure 3. Parameters α and β as functions of R_λ . These values are calculated by performing a least-squares fit to $d \ln E / d \ln k$ versus k/k_d (see Fig. 1). Points indicated by crosses correspond to measurements done in the range $k/k_d < 3$ for all runs in Table 1; diamonds correspond to $k/k_d > 6$ for runs 1 to 4.

work is a function of the skewness and the dissipation length. The skewness is believed to be independent of R_λ (Kerr 1985), and thus it is likely that the behaviour of β observed in Fig. 2 (b) is consistent with the derivation in Sirovich *et al.* (1994).

These values for α and β are consistent with those reported by Chen *et al.* (1993) obtained with a DNS with a dissipation range resolution comparable to run number three in Table 1, but with a lower Reynolds number. The results presented here, and particularly with reference to Fig. 3, might indicate that the exponential functional form (1) set forth for the energy spectrum in the dissipation range may not be the most appropriate guess for the near-dissipation range, i.e. $k/k_d < 4$. Solidly establishing these parameters as functions of R_λ will require next-generation computer capabilities. At this point, we are faced with the dilemma of having to choose between performing DNS with relatively high values for R_λ or having a relatively well-resolved dissipation range ($k_{\max}/k_d \gg 1$) at a relatively low Reynolds number. This study shows a definite trend, and it should be taken as an effort to shed some light on these issues by using state-of-the-art DNS.

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References

- Chen, S., Doolen, G. D., Herring, J. R., Kraichnan, R. H., Orszag, S. A. and She, Z. S. 1993 *Phys. Rev. Lett.* **70**, 3051.
- Domaradzki, J. A. 1992 *Phys. Fluids* **A4**, 2037.
- Foias, C., Manley, O. and Sirovich, L. 1990 *Phys. Fluids* **A2**, 464.
- Kida, S., Kraichnan, R. H., Rogallo, R. S., Waleffe, F. and Zhou, Y. 1992 *Proceedings of the Summer Program 1992, Center for Turbulence Research, Stanford University and NASA/AMES Research Center*.
- Kerr, R. 1985 *J. Fluid Mech.* **153**, 31.
- Kerr, R. 1990 *J. Fluid Mech.* **211**, 309.
- Kida, S. and Murakami, Y. 1987 *Phys. Fluids* **30**, 2030.
- Kolmogorov, A. N. 1941 *C. R. Acad. Sci. URSS* **30**, 301.
- Kraichnan, R. H. 1959 *J. Fluid Mech.* **5**, 497.
- Lohse, D. and Müller-Groeling, A. 1995 *Phys. Rev. Lett.* **74**, 1747.
- McComb, W. D. 1990 *The Physics of Fluid Turbulence*, Chap. 2. Clarendon Press, Oxford.
- Manley, O. P. 1992 *Phys. Fluids* **A4**, 1320.
- Novikov, E. A. 1961 *Dokl. Akad. Nauk. SSR* **139**, 331.
- Orszag, S. A. 1977 *Fluid Dynamics. Proceedings of 1973 Les Houches Summer School of Theoretical Physics* (ed. R. Balian and J. L. Peube), p. 237. Gordon and Breach, New York.
- Saddoughi, S. G. and Veeravalli, S. V. 1994 *J. Fluid Mech.* **268**, 333.
- Sanada, T. 1992 *Phys. Fluids* **A4**, 1086.
- Sirovich, L., Smith, L. and Yakhot, V. 1994 *Phys. Rev. Lett.* **72**, 344.
- Smith, L. M. and Reynolds, W. C. 1991 *Phys. Fluids* **A3**, 992.
- Sreenivasan, K. R. 1985 *J. Fluid Mech.* **151**, 81.
- Townsend, A. A. 1951 *Proc. R. Soc. Lond.* **A208**, 534.
- Zhou, Y. 1993 *Phys. Fluids* **A5**, 2511.