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Discontinuous Backstepping for Stabilization of Nonholonomic Mobile Robots

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Abstract

The paper presents a new method of performing integrator backstepping in systems that are discontinuous, either due to their inherent structure or because of the applied control input. The proposed technique is applied to the stabilization problem of the dynamic system of a nonholonomic mobile robot. Simulation studies indicate that the methodology can also help alleviate the problem of chattering that is commonly associated with discontinuous nonholonomic controllers.

1 Introduction

Stabilization of nonholonomic mobile robots has been a subject of intense research in the past years [1, 2, 3]. The implications of nonholonomic constraints on the kind of admissible control inputs for this class of systems has made the problem particularly challenging [4]. Many approaches have been proposed to address the issue of nonholonomic stabilization and can be broadly characterized as open loop strategies [5, 6], time-varying feedback designs [7, 8, 9] and discontinuous static feedback methods [10, 11, 12].

Although nonholonomic systems arise mainly from dynamic mechanical systems subject to preservation laws, the control problem has usually been tackled at the kinematic level, with few exceptions [13, 14, 15]. This is mainly due to the fact that the kinematic model captures their nonholonomic nature while abstracting away the details of their full dynamics. Implementation of a particular solution designed at the kinematic level has not been thought to be particularly difficult. However, this is not always the case.

A popular way of implementing a kinematic control law to a dynamic nonholonomic system is by backstepping [16] the velocity control commands to acceleration input [17, 18]. The technique can be applied only to continuous kinematic control laws, since the backstepping procedure requires their differentiation with respect to time. This constraint excludes an important class of nonholonomic controllers which are based on switching or introduce discontinuities.

This paper addresses the problem of backstepping a discontinuous nonholonomic kinematic controller into a dynamic equivalent one in order to stabilize a dynamic unicycle-type mobile robot. The approach is based on non smooth analysis [19] and relatively recent developments on application of Lyapunov stability to non smooth systems [20]. The results on discontinuous backstepping presented in this paper are not confined to the case of nonholonomic stabilization but can be applied to other discontinuous control systems. In fact, it turns out that backstepping the discontinuity is advantageous and alleviates the consequences of high frequency switching.

The paper is organized as follows. Section 2 gives a formal problem description of the issue addressed in this paper. Section 3 presents the design of a discontinuous state feedback controller that can globally asymptotically stabilize a unicycle-type mobile robot. In section 4 this discontinuous control law is backstepped through the feedback linearized dynamics of the mobile robot. Section 5 provides simulation evidence that support the theoretic developments and indicates the merits of the proposed technique. The results are summarized in section 6.

2 Problem description

Consider a nonholonomic mobile robot that moves on the horizontal plane. The coordinates of its position on the plane are given by (x, y), whereas its orientation with respect to an inertial reference frame is denoted by θ

Let the kinematics of the mobile robot be described by the equations of a unicycle:

$$\begin{cases} \dot{x} & = v\cos\theta \\ \dot{y} & = v\sin\theta \\ \dot{\theta} & = \omega \end{cases} \Leftrightarrow \quad \dot{\boldsymbol{q}} = \mathsf{G}\boldsymbol{u} \tag{1}$$

where $\boldsymbol{u} = (v, \omega)$ are the velocity control inputs.

Given the dynamic equations of the mobile robot:

$$\dot{\boldsymbol{q}} = \mathsf{G}\boldsymbol{u} \tag{2a}$$

$$\dot{\boldsymbol{u}} = \mathsf{M}(\boldsymbol{q})^{-1}(\boldsymbol{f} - \mathsf{R}(\boldsymbol{q}, \boldsymbol{u})) \tag{2b}$$

we seek a time invariant state feedback control law $\boldsymbol{f}(\boldsymbol{q})$ that stabilizes the mobile robot (2).

3 Discontinuous kinematic control

For self-containment, we first give some definitions:

Definition 1 ([19]). *Let f $f : X \to \mathbb{R}$ be Lipschitz near $x \in X$, where X is a Banach space, and let v be any vector in X. The generalized directional derivative of f at x in the direction v, denoted $f^\circ(x; v)$ is defined as follows:*

$$f^\circ(x; v) \triangleq \lim_{\substack{y\to x \\ t\to 0}} \sup \frac{f(y + tv) - f(y)}{t}$$

Definition 2 ([19]). *The generalized gradient of a locally Lipschitz $f : X \to mathbbR$ at x, denoted $\partial f(x)$, is the subset of the dual space X^* of continuous linear functionals on X:*

$$\partial f(x) \triangleq \{\zeta \in X^* \mid f^\circ(x; v) \geq \langle\zeta, v\rangle \forall v \in X\}$$

Theorem 1 ([20]). *Let $x(\cdot)$ be a Filippov solution to $\dot{x} = f(x, t)$ on an interval containing t and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and in addition, regular function. Then $V(t, x(t))$ is absolutely continuous, $\frac{d}{dt}V(t, x(t))$ exists almost everywhere and*

$$\frac{d}{dt}V(t, x(t)) \in^{\text{a.e.}} \dot{\tilde{V}}(t, x)$$

where

$$\dot{\tilde{V}}(t, x) \triangleq \bigcap_{\xi\in\partial V(t,x(t))} \xi^T \begin{pmatrix} F(t, x(t)) \\ 1 \end{pmatrix}$$

In fact, the globally Lipschitz continuity requirement of the above Theorem can be relaxed. Now, consider the kinematic equations of a unicycle type mobile robot (1). The following proposition presents a kinematic controller that renders (1) globally asymptotically stable. Consider the following definition for the signum function :

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Proposition 1. *The following feedback control law:*

$$v = \operatorname{sgn}(x)k_v[(y^2 - x^2)\cos\theta - 2xy\sin\theta] \tag{3a}$$

$$\omega = k_\omega(\arctan2(2xy, x^2 - y^2) - \theta) \tag{3b}$$

where k_v and k_ω are positive constants, asymptotically stabilizes (1) to the origin.

Proof. Consider the positive semidefinite functions:

$$V_1(x, y) = \frac{x^2 + y^2}{x}, \tag{4a}$$

for $(x, y, \theta) \in M^1 = \{(x, y, \theta)\,|x \geq 0\} \setminus \{\mathbf{0}\}$ and

$$V_2(x, y) = -\frac{x^2 + y^2}{x}, \tag{4b}$$

for $(x, y, \theta) \in M^2 = \{(x, y, \theta)\,|x < 0\}$. Note that $\{(x, y, \theta) \mid x = 0\} \notin M^1, M^2$ and thus V_1, V_2 are well defined. Then on each region M^1, M^2, (1) under control law (3) is asymptotically stable. In $M^1 \setminus \{(x, y, \theta)|x = 0\}$:

$$\begin{aligned} \dot{V}_1 &= -\frac{y^2 - x^2}{x^2}v\cos\theta + \frac{2xy}{x^2}v\sin\theta \\ &= -k_v(\frac{(y^2 - x^2)^2}{x^2}\cos^2\theta + 4y^2\sin^2\theta) \leq 0 \end{aligned}$$

Similarly, in $M^2 \setminus \{(x, y, \theta)|x = 0\}$:

$$\begin{aligned} \dot{V}_2 &= \frac{y^2 - x^2}{x^2}v\cos\theta - \frac{2xy}{x^2}v\sin\theta \\ &= -k_v(\frac{(y^2 - x^2)^2}{x^2}\cos^2\theta + 4y^2\sin^2\theta) \leq 0 \end{aligned}$$

In both cases equality is satisfied for $(x, y) = (0, 0)$ for all θ. For any invariant set in $(0, 0, \theta)$ it should be:

$$\omega = 0 \to \theta = 0$$

by definition of arctan2. Therefore, by LaSalle's invariant principle the system is asymptotically stable in the closures of both $M^1 \setminus \{(x, y, \theta)|x = 0\}$ and $M^2 \setminus \{(x, y, \theta)|x = 0\}$. To establish global asymptotic stability we need a certain transversallity condition, stated in the following Lemma:

Lemma 1 ([10]). *Let M^1, M^2 be two open connected subsets of $\mathbb{R}^n$ such that $M^1 \cup M^2 = \mathbb{R}^n \setminus \{0\}$. Let $f^i : M^i \to \mathbb{R}^n$, $i = 1, 2$ be two vector fields. Assume also, that there exists a separating hypersurface Γ with $0 \in \Gamma$ and $\Gamma \setminus \{0\} \subset M^1 \cap M^2$. Let C^i, C^2 be two connected components of $\mathbb{R}^n \setminus \Gamma$ and assume that $C^i \subset M^i$ and that f^i points towards C^i on Γ for $i = 1, 2$. Finally assume that f^1, f^2 are asymptotically stable on M^1, M^2. Then, the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$f(x) = \begin{cases} f^1(x) & \text{if } x \in (\Gamma \setminus \{0\}) \cup C^1 \\ f^2(x) & \text{if } x \in C^2 \\ 0 & \text{if } x = 0 \end{cases}$$

is globally asymptotically stable.

Then, a direct calculation can show that the transversllity condition is satisfied. Therefore (1) is globally asymptotically stable. □

Remark 1. *Convergence of θ to θ_d can be made exponentially fast by the choice:*

$$\omega = k_\omega[\arctan2(2xy, x^2 - y^2) - \theta] + \frac{2v(x \sin\theta - y\cos\theta)}{x^2 + y^2}$$

which implies that $\dot{\theta} - \dot{\theta}_d = -k_\omega(\theta - \theta_d)$.

Lyapunov's direct method has been extended to non smooth systems [20]. The following result formalizes this extension:

Theorem 2 ([20]). *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ be essentially locally bounded and $\mathbf{0} \in K[\mathbf{f}](\mathbf{0}, t)$ in a region $Q \supset \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\| < r\} \times \{t | t_0 \le t < \infty\}$. Also, let $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a regular function satisfying*

$$V(\mathbf{0}, t) = 0$$

and

$$0 < V_1(\|\mathbf{x}\|) \le V(\mathbf{x}, t) \le V_2(\|\mathbf{x}\|), \text{ for } x \ne 0$$

in Q for some $V_1, V_2 \in$ class $\mathcal{K}$. Then,

1. *$\dot{\tilde{V}} \le 0$ in Q implies $\mathbf{x} \equiv \mathbf{0}$ is a uniformly stable solution.*

2. *If in addition, there exists a class $\mathcal{K}$ function $\omega(\cdot)$ in Q with the property*

$$\dot{\tilde{V}}(\mathbf{x}, t) \le -\omega(\mathbf{x}) < 0$$

then the solution $\mathbf{x}(t) \equiv \mathbf{0}$ is uniformly asymptotically stable.

4 Backstepping discontinuous inputs

Let us turn out attention now to the dynamic model of the mobile robot (2). It follows that the stabilizing control input (3) can no longer applied since $\boldsymbol{u}$ is now a state variable.

The feedback input transformation:

$$\boldsymbol{f} = \mathsf{R}(\boldsymbol{q}, \boldsymbol{u}) + \mathsf{M}(\boldsymbol{q})\boldsymbol{v} \tag{5}$$

linearizes the lower part of (2):

$$\dot{\boldsymbol{q}} = \mathsf{G}\boldsymbol{u} \tag{6}$$

$$\dot{\boldsymbol{u}} = \boldsymbol{v} \tag{7}$$

where $\boldsymbol{v}$ is now the new control input. If (3) was smooth, then by backstepping it through the integrators (7) one could stabilize the complete dynamic system (2).

However, the kinematic control law (3) is discontinuous to conform with Brockett's condition [4]. This motivated the extension of the technique of integrator backstepping to handle the case where the original control law that stabilizes the subsystem is non smooth. The result obtained can be used in general for systems that are described by discontinuous ordinary differential equations:

Theorem 3. *Consider the system:*

$$\dot{\boldsymbol{\eta}} = \boldsymbol{f}(\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{\xi} \tag{8}$$

$$\dot{\boldsymbol{\xi}} = \boldsymbol{u} \tag{9}$$

where $\boldsymbol{\eta} \in \mathbb{R}^n$, $\boldsymbol{\xi} \in \mathbb{R}^m$. Assume that the subsystem (8) can be stabilized by a control law $\boldsymbol{\xi} = \boldsymbol{\phi}(\boldsymbol{\eta})$ with $\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0}$, and that there is a regular (possibly non smooth) Lyapunov function $V(\boldsymbol{\eta})$ for which it holds:

$$W(\boldsymbol{\eta}) \triangleq -\bigcap_{\boldsymbol{\lambda} \in \partial V(\boldsymbol{\eta})} \boldsymbol{\lambda}^T K[\boldsymbol{f}(\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{\phi}(\boldsymbol{\eta})] > 0 \tag{10}$$

where $K[\cdot]$ is the Filippov solution. Then the following control law (dependence of $\boldsymbol{g}$ and $\boldsymbol{\phi}$ on $\boldsymbol{\eta}$ is dropped):

$$\boldsymbol{u} = -\text{diag}\left\{\|\boldsymbol{\xi} - \boldsymbol{\phi}\|_2^{-2} \, V^\circ(\boldsymbol{\eta}; \boldsymbol{g}(\boldsymbol{\xi} - \boldsymbol{\phi})) + \boldsymbol{k}_z\right\} \cdot (\boldsymbol{\xi} - \boldsymbol{\phi}(\boldsymbol{\eta})) + \dot{\tilde{\boldsymbol{\phi}}} \tag{11}$$

where $V^\circ(\cdot)$ is the generalized derivative [19], $\dot{\tilde{\boldsymbol{\phi}}}$ is the generalized time derivative [20] and $\boldsymbol{k}_z$ a positive constant gain vector, stabilizes asymptotically the system (8)-(9) to the origin.

Proof. The proof structure is adopted from [21]. By a change of variables:

$$\boldsymbol{z} = \boldsymbol{\xi} - \boldsymbol{\phi}(\boldsymbol{\eta})$$

the system (8)-(9) can be written as:

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= f(\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{\phi}(\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{z} \\ \dot{\boldsymbol{z}} &= \boldsymbol{u} - \dot{\boldsymbol{\phi}}\end{aligned}$$

Taking $\boldsymbol{v} = \boldsymbol{u} - \dot{\boldsymbol{\phi}}$:

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \boldsymbol{f}(\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{\phi}(\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{z} \\ \dot{\boldsymbol{z}} &= \boldsymbol{v}\end{aligned}$$

Consider the following Lyapunov function:

$$V_a(\boldsymbol{\eta}, \boldsymbol{\xi}) \triangleq V(\boldsymbol{\eta}) + \frac{1}{2}\boldsymbol{z}^T\boldsymbol{z}$$

Its time derivative is

$$\dot{\tilde{V}}_a = -W(\boldsymbol{\eta}) + \bigcap_{\boldsymbol{\lambda} \in \partial V(\boldsymbol{\eta})} \boldsymbol{\lambda}^T \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{z} + \boldsymbol{z}^T\boldsymbol{v}$$

Let $\boldsymbol{v}$ be defined as follows:

$$\boldsymbol{v} \triangleq -\mathrm{diag}\left\{\|\boldsymbol{z}\|_2^{-2} V^\circ(\boldsymbol{\eta}; \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{z}) + \boldsymbol{k}_z\right\}\boldsymbol{z}$$

where $\boldsymbol{k}_z$ is a positive constant gain vector. Then, from the definition of generalized derivative it follows:

$$\bigcap_{\boldsymbol{\lambda} \in \partial V(\boldsymbol{\eta})} \boldsymbol{\lambda}^T \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{z} - V^\circ(\boldsymbol{\eta}; \boldsymbol{g}(\boldsymbol{\eta})\boldsymbol{z}) \leq 0,$$

and $\dot{\tilde{V}}_a$ becomes negative definite. Application of Theorem 2 completes the proof. □

According to Theorem 3 at the discontinuity point the control input is no longer a real-valued vector, but instead it is a set, as indicated by $\dot{\tilde{\boldsymbol{\phi}}}(\boldsymbol{\eta})$. Any value within that set is admissible in the sense that asymptotic stability is preserved. If the original control is a switching - sliding mode type law, then zero is always contained in the set and can be selected to substitute $\tilde{\boldsymbol{\phi}}(\boldsymbol{\eta})$ in (11).

5 Simulation results

The technique is tested on the system:

$$\begin{aligned}\dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \\ \dot{v} &= u_1 \\ \dot{\omega} &= u_2\end{aligned}$$

that describes the linearized dynamics of a nonholonomic mobile robot. Results are depicted in Figures 1 - 6 for several initial conditions. The following values for the controller gains were used:

$$k_v = 3 \qquad k_\omega = 3 \qquad \boldsymbol{k}_z = \begin{bmatrix}10 & 1\end{bmatrix}$$

In case 1 the initial conditions are $x(0) = 0$, $y(0) = 1$ and $\theta = \frac{\pi}{2}$. Figure 1 shows the path traversed by the mobile robot and Figure 2 depicts the evolution of its configuration variables. This case shows that the separating surface Γ in Lemma 1 does not introduce any singularities despite the fact that the Lyapunov functions (4) are not defined there. For case 2 initial conditions are $x(0) = -1$, $y(0) = 0$ and $\theta = \frac{\pi}{2}$. This is another suspicious case for potential singularities, the controllers however are well behaved. The resulting path is shown on Figure 3 and the time histories of the configuration variables in Figure 4.

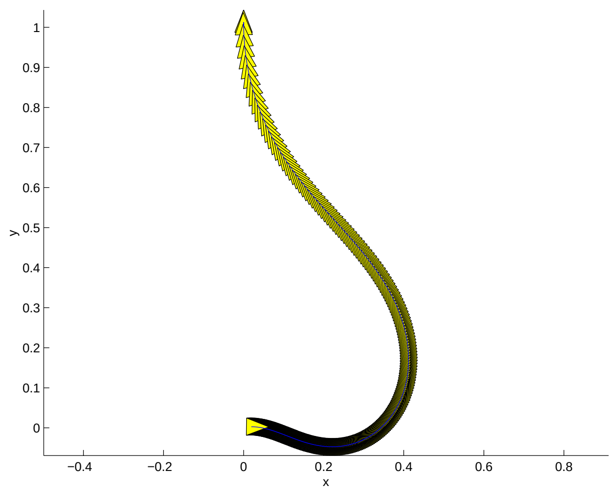

Figure 1: Path to the origin in case 1

Figures 5 - 6 show that besides stabilization, backstepping discontinuous input can lead to alleviation of chattering. Initial conditions for this case are $x = 1$, $y = 1$, $\theta = \frac{3\pi}{4}$. The integrator through which backstepping is performed acts as a low pass filter for the switching input (Figure 5). The bandwidth of this “filter” is adjusted by the gains of the backstepping controller: higher gains ensure close tracking of the reference control input (including discontinuities); lower gains make the system less sensitive to switching but reduce convergence rate (Figure 6).

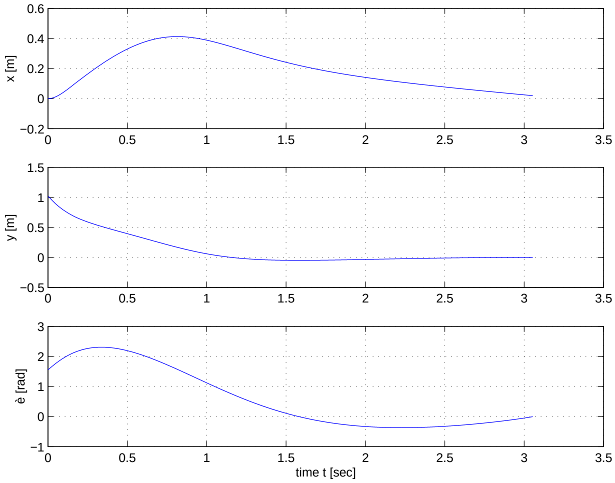

Figure 2: Trajectories of configuration variables in case 1

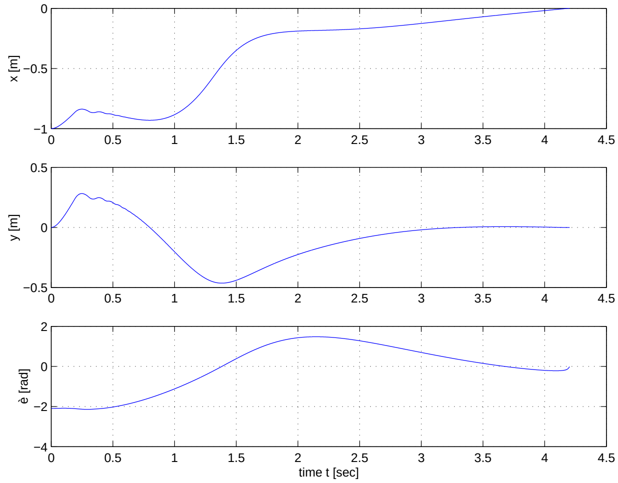

Figure 4: Trajectories of configuration variables in case 2

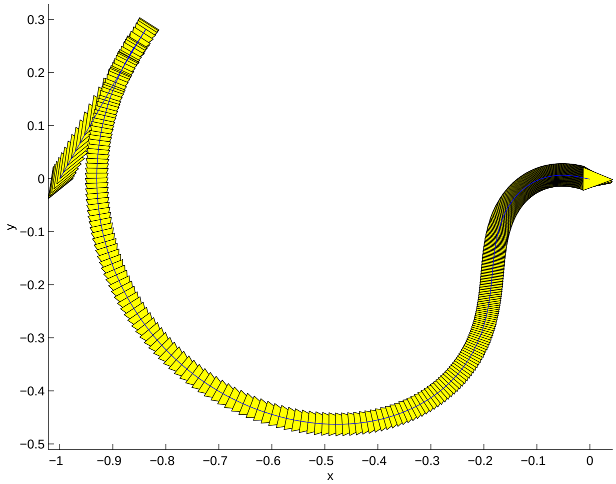

Figure 3: Path to the origin in case 2

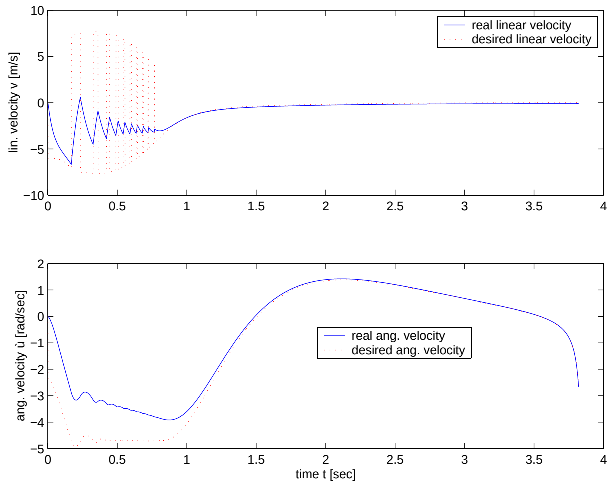

Figure 5: Filtering of chattering through backstepping

6 Conclusions

The paper presents a novel technique for backstepping discontinuous control signals with application to stabilization for the dynamic model of a nonholonomic mobile robot. The methodology is not restricted, though, to nonholonomic systems but can be applied to a broad class of strictly feedback discontinuous nonlinear systems. It is also indicated that backstepping discontinuous reference control signals can help alleviate the effect of chattering. Future research directions include generalization of the method to other, more complicated forms of backstepping.

References

[1] Z. Li and J.F. Canny, *Nonholonomic Motion Planning*, Kluwer Academic Publishers, 1993.

[2] R.M. Murray, Z. Li, and S.S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, CRC Press, 1994.

[3] I. Kolmanovsky and N.H. McClamroch, "Developments in nonholonomic control problems," *IEEE Control Systems*, pp. 20–36, December 1995.

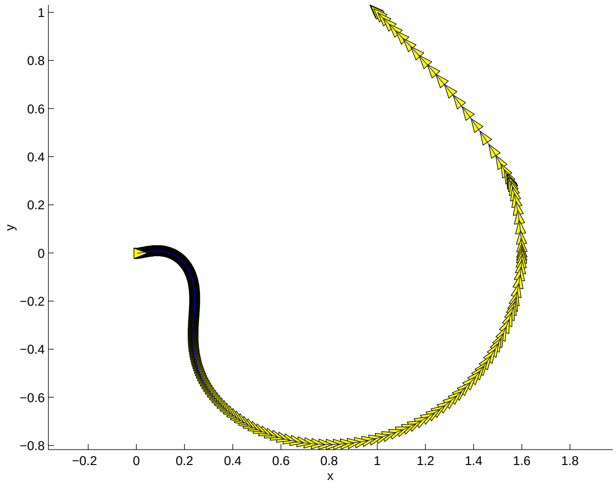

Figure 6: Path to the origin under chattering in reference input

[4] R. Brockett, *Control Theory and Singular Riemannian Geometry*, New Directions in Applied Mathematics. Springer, 1981.

[5] G.A. Lafferriere and H. Sussmann, "A differential geometric approach to motion planning," in *Nonholonomic Motion Planning*, Z. Li and J.F. Canny, Eds., pp. 235–270. Kluwer Academic Publishers, 1993.

[6] R.W. Brockett and L. Dai, "Nonholonomic kinematics and the role of elliptic functions in constructive controllability," in *Nonholonomic Motion Planning*, Z. Li and J.F. Canny, Eds., pp. 1–22. Kluwer Academic Publishers, 1993.

[7] J.-P. Pomet, "Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift," *Systems & Control Letters*, vol. 18, pp. 147–158, 1992.

[8] R.M. Murray and S.S. Sastry, "Nonholonomic motion planning : Steering using sinusoids," *IEEE Transactions on Automatic Control*, pp. 700–716, May 1993.

[9] R.T. M' Closkey and R.M. Murray, "Exponential stabilization of driftless nonlinear control systems using homogeneous feedback," *IEEE Transactions on Robotics and Automation*, vol. 42, no. 5, pp. 614–628, 1997.

[10] G.A. Lafferriere and E.D. Sontag, "Remarks on control lyapunov functions for discontinuous stabilizing feedback," in *Proceedings of the 32nd IEEE Int. Conf. on Decision and Control*, San Antonio, TX, 1993, pp. 2398–2403.

[11] C. Canudas de Wit and O.J. Sordalen, "Exponential stabilization of mobile robots with nonholonomic constraints," *IEEE Transactions on Automatic Control*, vol. 13, no. 11, pp. 1791–1797, 1992.

[12] A. Astolfi, "Discontinuous control of nonholonomic systems," *Systems & Control Letters*, vol. 27, pp. 37–45, 1996.

[13] G. Campion, B. d'Andrea Novel, and G. Bastin, "Modelling and state feedback control of nonholonomic mechanical systems," in *Proceedings of the 1991 IEEE Conference on Decision and Control*, Brighton, England, Dec. 1991, pp. 1184–1189.

[14] George J. Pappas and K. J. Kyriakopoulos, "Modeling and feedback control of nonholonomic mobile vehicles," in *Proceedings of the 1992 IEEE Conference in Decision and Control*, Dec. 1992, pp. 241–244.

[15] A. M. Bloch M. Reyhanoglu and N. H. McClamroch, "Control and stabilization of nonholonomic dynamic systems," *IEEE Transactions on Automatic Control*, pp. 1746–1757, November 1992.

[16] Miroslav Krstić, Ioannis Kanellakopoulos, and Petar Kokotović, *Nonlinear and Adaptive Control Design*, John Willey and Sons, 1995.

[17] R. Fierro and F.L. Lewis, "Control of a nonholonomic mobile robot: backstepping kinematics into dynamics," in *Proc. of 34th IEEE Conference on Decision and Control*, New Orelans, LA, 1995, pp. 3805–3810.

[18] Zhong-Ping Jiang and Henk Nijmeijer, "Tracking control of mobile robots: A case study in backstepping," *Automatica*, vol. 33, no. 7, pp. 1393–1399, 1998.

[19] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Addison-Wesley Publishers, 1983.

[20] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1910–1914, 1994.

[21] K. Khalil, Hassan, *Nonlinear Systems*, Prentice Hall, 1996.