Discrete Asymptotic Abstractions of Hybrid Systems

Jorge L. Piovesan, Herbert G. Tanner, and Chaouki T. Abdallah

Abstract— **In this paper we introduce the notion of Finite Time Mode Abstraction to relate a hybrid automaton to a timed automaton that preserves the stability and reachability properties of the former. The abstraction procedure discards the continuous dynamics of each mode in the hybrid automaton completely, keeping only the information about the maximum time in which the continuous state makes a discrete jump. This information is used to construct a timed automaton, based on the original hybrid automaton, and to prove that the stability and reachability properties of the original system are retained in the abstract timed automaton. In the process of abstracting a hybrid to a timed automaton we introduce a new notion of hybrid distance metric, which provides information about both the number of discrete transitions that a system would have to make to go from one hybrid state to another, and the distance between the continuous parts of such hybrid states.**

I. Introduction

Complex nonlinear systems with a large number of degrees of freedom are notoriously difficult to analyze. They usually do not lend themselves to common control design methodologies. This is especially true with hybrid systems, where the interaction between the discrete and continuous dynamics makes the analysis task formidable, even for simple cases. In synchronization tasks, or rendezvous problems involving multiple autonomous systems, coordinating may require reliable timing information, but exact position history may be irrelevant. The question that arises is whether some system details not related to the problem at hand can be safely ignored, and enable a more efficient solution.

One school of thought that envisions managing the complexity of such tasks advocates system abstraction, as a is the selective retention of information pertinent to a specific task or objective. It is a concept used widely in computer science, formally described in terms of a *bisimulation* relation [10]. In [13]–[15], (purely) continuous systems are related to each other in terms of their vector fields: the vector field of the quotient system is the image of that of the original system under a surjective (Φ) map. The link between this form of abstraction and the notion of bisimulation is made clearly (for the linear case) in [12] and (for the nonlinear case) in [17], [20]. In [18], where abstraction of nonlinear control systems was rephrased in a categorical framework, building upon the differential geometric interpretation of bisimulation.

Bisimulation, however, may be too restrictive at times. The survey paper [1] demonstrates that in order to obtain bisimulations for hybrid systems in general, one has to restrict either the discrete logic that governs the transitions, or the type of continuous dynamics. Certain undecidability results are presented [1] to indicate the limits of abstraction based on bisimulation. Such results motivate less restrictive conditions, posed by *simulation* relations [10]. In such cases, one may choose to abandon the search for input-output equivalence for the hope of obtaining some property inclusion. Abstractions that are based on simulation relations, were obtained in [19] for linear systems, in a similar framework as that of [12]. When a system is abstracted by means of a simulation relation, the abstract system will generally have a richer behavior (through the abstraction map), but if a property is verified for the abstract system, it holds for the original.

Both these abstraction methods, having their mechanisms based on a different type of equivalence relation between pairs of states, are similar in the sense that in the continuous world, they associate one vector field with another. There could be cases where even this is too restrictive, or unnecessary. The motivation for the work presented here comes from the desire to devise a consistent method for mapping continuous (or hybrid) dynamics into (almost purely) discrete ones, in a meaningful way. We attempt to characterize the asymptotic behavior of a system, rather than its local direction of motion, by ensuring that after a certain time period, initial system states in set A have "collapsed" into points in a set B. We are interested in *where* a system will *eventually* end up, and under certain assumptions, we are willing to sacrifice knowing *exactly how* it will go there. These ideas are then exploited to map a hybrid automaton to a timed-automaton. The only information that is preserved in the later model of computation is the identity of the continuous partitions and the (maximum) time required by the system to reach one from another. The assumptions that we make in order to safely ignore the "transient" phase could increase the number of discrete states. We feel, however, that this is a reasonable price to pay for obtaining a discrete representation of the system dynamics. The stricter assumption we make on the (time-invariant) continuous part is the existence of a finite number of disjoint limit sets.

II. Continuous Dynamical Systems

Let M be a Banach manifold modeled on $\mathbb{R}^n$. A standard definition of a (closed loop) time invariant system in terms of a (smooth) vector field $f(x)$ on M is:

Definition 1 ([5]) *A system consists of a pair (M, f) where M is a manifold and $f : M \to TM$ a smooth vector field.*

Let $\Phi_t(p)$ be the flow [4] of the vector field f; that is, $\forall p \in M$, $\forall t > 0$, $\Phi_t(p) = \sigma_p(t)$. We assume that flows are

Jorge Piovesan and Chaouki Abdallah are with the Department of Electrical and Computer Engineering, of the University of New Mexico, Albuquerque NM 87131. Herbert Tanner is with the Department of Mechanical Engineering of the University of New Mexico, Albuquerque NM 87131.

ultimately bounded in the following sense:

Assumption 1 *For all* $p \in M$*, and for all* $t \geq 0$, $\sup_{t\geq 0} \|\Phi_t(p)\| < \infty$.

The norm $\|\cdot\|$ on M is assumed to be the one induced in M by a typical norm on $\mathbb{R}^n$. We first recall the concept of the positive limit set of the trajectories of a system (M, f) [7]:

Definition 2 (Positive limit set) *Let* $\Phi_t(p)$ *be a flow of the system* (M, f) *starting from* $p \in M$*. Then* $q \in M$ *is said to be a positive limit point of* $\Phi_t(p)$ *if there is a sequence* $\{t_n\}$*, with* $t_n \to \infty$ *as* $n \to \infty$*, such that* $\Phi_{t_n}(p) \to q$ *as* $n \to \infty$*. The set of all limit points of* $\Phi_t(p)$*,* $\forall p \in M$ *is called the positive limit set of* Φ_t*.*

We define the distance of a point to a set as [7]:

Definition 3 (Distance to a subset of M) *The distance of a point* p *of a Banach manifold* M *to a subset* $Z \subset M$ *is denoted* $\operatorname{dist}(p, Z)$ *and is defined as* $\operatorname{dist}(p, Z) \triangleq \inf_{z\in Z} \|p - z\|$.

III. Hybrid Systems

A hybrid system is a type of system in which continuous time and discrete event dynamics blend together to enrich its behavior. One of the most common forms of representing a hybrid system is the hybrid automaton [2], [9]:

Definition 4 (Hybrid Automaton) *A hybrid automaton* H *is a collection* $H = (Q, X, f, \mathrm{I}nit, D, E, G, R)$ *where,*

- Q *is a finite set of discrete variables.*
- X *is a finite dimensional set of continuous variables.*
- $f : \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$ *is a vector field.*
- $\mathrm{I}nit \subseteq \mathbf{Q} \times \mathbf{X}$ *is a set of initial states.*
- $D : \mathbf{Q} \to P(\mathbf{X})$ *is a domain.*
- $E \subseteq \mathbf{Q} \times \mathbf{Q}$ *set of set of edges.*
- $G : E \to P(\mathbf{X})$ *is the guard condition.*
- $R : E \times \mathbf{X} \to P(\mathbf{X})$ *is the reset map.*

where $\mathbf{Q}$ *denotes the set of al possible valuations of* $q \in Q$*,* $\mathbf{X}$ *denotes a smooth manifold for* X*,* $T\mathbf{X}$ *denotes the tangent bundle of* $\mathbf{X}$ *and* $P(\mathbf{X})$ *is the power set of* $\mathbf{X}$*.* $(q, x_q) \in \mathbf{Q} \times \mathbf{X}$ *is referred as the state* h *of the hybrid automaton* H*.*

We study a subset of the hybrid automata of Definition 4. Similarly to [9], we assume:

Assumption 2 *Consider hybrid automata as in 4 for which:*

- X *is subset of a Banach manifold* M*, modeled on* $\mathbb{R}^n$*;*
- $G(e) \neq \emptyset$*,* $\forall e \in E$*;*
- $R(e, x) \neq \emptyset$*,* $\forall x \in G(e)$*;*
- *For each* $q \in Q$*, the positive limit set* L^+ *of the flows* $f(q, x)$ *satisfies* $L^+(q) \subseteq G(e)$*, for* $e \in \{(q, p) \mid (q, p) \in E\}$*.*

The last condition implies that the positive limit sets of the flows are contained in the guards. Note that we do not assume global Lipschitz continuity of f; instead, we use the boundedness condition of Assumption 1, which also ensures the existence of a positive limit set L^+ for the flows of f.

The positive limit set of Φ_t for a given $q \in Q$, L^+, may be disconnected. For a given discrete state $q \in Q$, let $L_i^+(q)$ $i = 1, \ldots, \ell$ be a disconnected component of $L^+(q)$. We assume that $\ell < \infty$, considering the verification of this condition a control design issue to be addressed in the future. Whenever a domain D contains multiple disconnected components of $L^+(q)$, (and given that each component belongs to a different guard,) we partition the given D into regions that have a single, common component L_i^+ as shown in Figure 1.

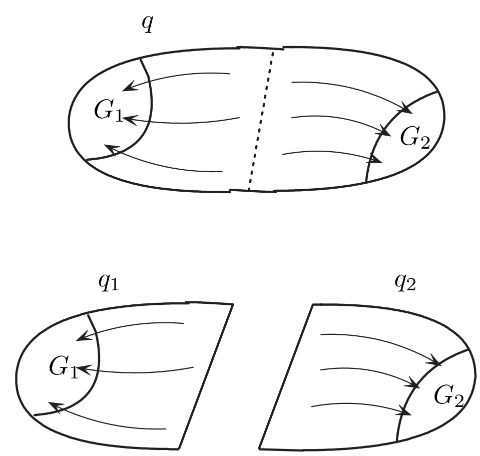

Fig. 1. A domain is partitioned according to the inclusion of connected components of the positive limit set within the guards.

This refinement also guarantees that the flows of $f(q, x)$ in $D(q)$ do not activate any other guards before reaching the one where L^+ is contained. For the rest of the paper the following notation is used. The labels of different modes are denoted as subscripts, i.e., v_a. The instantaneous values of the discrete sequences of the hybrid system are denoted as square brackets, i.e., $v[i]$. The continuous time evolution is denoted in parentheses, as in $v(t)$. Finally, v_q denotes the state of that variable at the beginning of an active period of the mode, $q \in Q$, and v'_q denotes the state of that variable at the end of an active period of the mode $q \in Q$.

Definition 5 (Hybrid time trajectory [8]) *A hybrid time trajectory is a finite or infinite sequence of intervals* $\tau = \{I[i]\}_{i=0}^N$*, such that*

- $I[i] = [\tau[i], \tau'[i]]$*, for all* $i < N$*;*
- *if* $N < \infty$*, then either* $I[N] = [\tau[N], \tau'[N]]$*, or* $I[N] = [\tau[N], \tau'[N])$*;*
- $\tau[i] \leq \tau'[i] = \tau[i + 1]$ *for all* i*.*

where $\tau[i]$ *are the times at which the discrete transition from the mode* $q[i - 1]$ *to* $q[i]$ *takes place.*

The set $\langle\tau\rangle \triangleq \{1, 2, ..., N\}$ if N is finite and $\{1, 2, ...\}$ if $N = \infty$. We define $|\tau| = \sum_{i\in\langle\tau\rangle}(\tau'[i] - \tau[i])$.

Definition 6 (Execution [8]) *An hybrid automaton execution is a triple* $\chi = (\tau, q, x)$*, with* τ *a hybrid time trajectory,*

$q : \langle\tau\rangle \to Q$ a map, and $x = \{x[i] : i \in \langle\tau\rangle\}$ a collection of differentiable maps $x[i] : I[i] \to X$, such that

- $(q[0], x[0](0)) \in \mathrm{I}nit$
- *for all $t \in [\tau[i], \tau'[i])$, $\dot{x}[i](t) = f\big(q[i], x[i](t)\big)$ and $x[i](t) \in D(q[i])$;*
- *for all $i \in \langle\tau\rangle\backslash\{N\}$, $e = (q[i], q[i+1]) \in E$, $x[i](\tau'[i]) \in G(e)$, and $x[i+1](\tau[i+1]) \in R(e, x[i](\tau'[i]))$.*

We use $(q_0, x_0) = (q[0], x[0](0))$ to denote the initial condition, $\epsilon_H(q_0, x_0)$ to denote the set of all executions of H with initial condition $(q_0, x_0) \in \mathrm{I}nit$, $\epsilon_H^*(q_0, x_0)$ the set of all finite executions of H with $(q_0, x_0) \in \mathrm{I}nit$, and $\epsilon_H^\infty(q_0, x_0)$ the set of all the infinite executions with $(q_0, x_0) \in \mathrm{I}nit$.

Definition 7 (Reachable Set [8]) *A hybrid state $h \in Reach(h_0)$ if there exists at least one finite execution $\epsilon_H^*(h_0)$ mapping h_0 to h. The set of all the hybrid states that can be reached from any initial condition is $Reach_H = \bigcup_{h_0 \in \mathrm{I}nit}(Reach(h_0))$*

A timed automaton is defined here as follows:

Definition 8 (Timed Automaton [1]) *A Timed Automaton is a hybrid automaton that satisfies the following properties:*

- *For every discrete variable $q \in Q$ the set $\mathrm{I}nit(q)$ is empty or a singleton, the set $D(q)$ is a rectangular set and the continuous flow is given by $f(q, x) = 1$*
- *For each edge $e \in E$ the set $G(e)$ is a rectangular set.*
- *For every edge $e \in E$ and for all $x \in X$, $R(e, x) = \{y \in X | y = x$ or $y = c$, where c is a constant vector.*

We finalize this section noting that a hybrid automaton can be represented by a directed graph [8], such that each discrete mode in Q is mapped to a vertex, which will contain the label of the mode, its domain and its continuous flow equation. Similarly, each edge, that represents a discrete transition will have a guard and a reset function attached to it (For an example see Figure 2). The directed graph related to the hybrid automaton H will be denoted as $\mathcal{G}_H$.

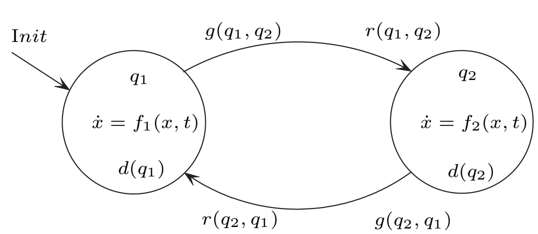

Fig. 2. Example of a Hybrid automaton represented using a graph.

IV. A New Hybrid Metric

We define in this section a notion of a hybrid distance that provides information about both the continuous and the discrete distances between two hybrid states. Since our graph is directly associated with a Hybrid Automaton H, we identify its nodes with the modes of H, which represent a distinct behavior of the underlying dynamical system.

Definition 9 (Discrete Distance) *Let the distance between two discrete states of a hybrid system q_1 and q_2 be the length of the shortest path[1] from mode q_1 to mode q_2 in the directed graph $\mathcal{G}_H$, associated with the Hybrid Automaton H. This distance is denoted by $d_D(q_1, q_2)$.*

Definition 10 *Let $A = A(\mathcal{G}_H)$ be the adjacency matrix* of *the directed graph $\mathcal{G}_H$ associated with H. The entries of A have their rows and columns indexed by the pair $(q_i, q_j) \in$ $\mathbf{Q} \times \mathbf{Q}$. Each entry (q_i, q_j) will be 1 when a transition is possible from q_i to q_j (an edge exists) and 0 otherwise.*

The adjacency matrix has the property that its r power will give as an entry at position (q_i, q_j) the number of directed paths from q_i to q_j of length r [6]. Based on this property we propose a procedure to calculate the discrete distance between to discrete modes in a hybrid automaton H:

Lemma 1 *The discrete distance $d_D(q_1, q_2)$ can be calculated as follows:*

$$d_D(q_1, q_2) = \begin{cases} \min_{r \in \mathbb{N}}\{r : (A^r)_{(q_1,q_2)} \neq 0\} & q_2 \in Reach(q_1) \\ \infty & \text{otherwise} \end{cases} \quad (1)$$

Proof: The discrete distance between q_1 and q_2 is the length of the shortest path from q_1 to q_2 in $\mathcal{G}_H$. Since $(A^r)_{(q_1,q_2)}$ gives the number of paths from q_1 to q_2 with length r ([6]-Lemma 8.1.2), then the shortest path from q_1 to q_2 is the minimum r that makes $(A^r)_{(q_1,q_2)} \neq 0$ whenever q_2 is reachable from q_1. If q_2 is not reachable, the distance is infinite by default. ■

Definition 11 (Hybrid Distance) *Let the distance between two hybrid states be $d_H(h_1, h_2) = \tanh(\|x_1 - x_2\|) + d_D(q_1, q_2)$, where $h_i = (q_i, x_i)$ for $i = 1, 2$ and $\|.\|$ is the norm on X.*

Using the $\tanh(\cdot)$ function of the norm in the distance expression gives different weight to the discrete part of the hybrid state; (hybrid) states in different discrete modes are considered to be much further apart than any continuous states in the same mode. In what follows, we show that the proposed function can serve as a metric on the space $Q \times X$, with the exception of symmetry: the existence of a path from q_1 to q_2 does not imply the existence of a path of the same length from q_2 to q_1. This distinction is not made in the related constructions found in [9], [16].

Proposition 1 *The hybrid distance $d_H(h_1, h_2)$ is zero if and only if $q_1 = q_2$ and $x_1 = x_2$.*

Proof: First note that the continuous portion of the hybrid distance $\tanh(\|x_1 - x_2\|)$ will only be zero when the argument of $\tanh(.)$ is zero and this will happen only when

[1]For a definiton of a path, see [6].

$x_1 = x_2$. Second note that, by definition 9, the discrete part of the hybrid distance $d_D(q_1, q_2)$ will be zero only when $q_1 = q_2$ which proves the proposition. ■

Proposition 2 *The hybrid distance $d_H(h_1, h_2) \geq 0$ for all q_1, q_2, x_1, and x_2.*

Proof: The $\tanh(\cdot)$ function is positive for positive arguments and zero if the argument is null. Since $\|x_1 - x_2\|$ is positive for all $x_1 \neq x_2$ and zero for $x_1 = x_2$ then $\tanh(\|x_1 - x_2\|)$ will be positive for all $x_1 \neq x_2$ and zero for $x_1 = x_2$. On the discrete part of the hybrid distance r represents the number of jumps that an state would have to take to reach another state. Since this variable is always nonnegative, and zero only for $q_1 = q_2$, $d_D(q_1, q_2)$ will always be nonnegative proving the proposition. ■

Proposition 3 *The hybrid distance $d_H(h_1, h_2)$ satisfies the triangle inequality $d_H(h_1, h_3) \leq d_H(h_1, h_2) + d_H(h_2, h_3)$ for all q_1, q_2, q_3, x_1, x_2, and x_3.*

To prove the above we will need the following Lemmas:

Lemma 2 *$d_D(q_1, q_3) \leq d_D(q_1, q_2) + d_D(q_2, q_3)$ for all q_1, q_2, and q_3.*

Proof: Consider a directed graph that contains $q_1, q_2, q_3 \in Q$ and analyze three cases:

a) I: f $q_1 = q_3$, then $d_D(q_1, q_3) = 0$ by proposition 1. Moreover it has been proven (proposition 2) that for every pair of modes $q_m, q_n \in Q$ the distance $d_D(q_m, q_n) \geq 0$. So $d_D(q_1, q_3) = 0 \leq d_D(q_1, q_2) + d_D(q_2, q_3)$.

b) I: f $q_1 \neq q_3$ and $q_3 \notin \text{Reach}(q_1)$ then $d_D(q_1, q_3) = \infty$, because there does not exist any path from q_1 to q_3. This implies that there will not exist any path between at least one of the pairs q_1, q_2 or q_2, q_3 causing at least one of the distances $d_D(q_1, q_2)$ or $d_D(q_2, q_3)$ to be infinite. Thus $d_D(q_1, q_3) = \infty = d_D(q_1, q_2) + d_D(q_2, q_3)$.

c) I: f $q_1 \neq q_3$ and $q_3 \in \text{Reach}(q_1)$ then $d_D(q_1, q_3) < \infty$. So assume without loss of generality that $q_3 \in \text{Reach}(q_2)$ and $q_2 \in \text{Reach}(q_1)$ (If any of this conditions is not satisfied then the lemma is trivially satisfied because at least one of the distances in the right hand side of the inequality would be infinite). Note that the minimum number of transitions to go from q_i to q_j for all $i = 1, 2$ and $j = 2, 3 : i \neq j$ is given by $d_D(q_i, q_j)$. So if the minimum path from q_1 to q_3 included q_2 then $d_D(q_1, q_3) = d_D(q_1, q_2) + d_D(q_2, q_3)$. Otherwise, if the minimum path between q_1 and q_3 did not include q_2 then moving the discrete state from q_1 through q_2 to q_3 would create a path with more jumps than going directly from q_1 to q_3, i.e. $d_D(q_1, q_3) < d_D(q_1, q_2) + d_D(q_2, q_3)$.

These three cases together prove that $d_D(q_1, q_3) \leq d_D(q_1, q_2) + d_D(q_2, q_3)$ for every $q_1, q_2, q_3 \in Q$. ■

Lemma 3 *$\tanh(\|x_1 - x_3\|) \leq \tanh(\|x_1 - x_2\|) + \tanh(\|x_2 - x_3\|)$ for all x_1, x_2, and x_3.*

Proof: It follows directly from the properties of the $\tanh(\cdot)$ function. ■

We now prove Proposition 3:

Proof: The triangle inequality in Proposition 3 can be rewritten as follows $\tanh(\|x_1 - x_3\|) + d_D(q_1, q_3) \leq \tanh(\|x_1 - x_2\|) + d_D(q_1, q_2) + \ldots + \tanh(\|x_2 - x_3\|) + d_D(q_2, q_3)$. Note that if $a \leq b$ and $c \leq d$ then $a + c \leq b + d$. Thus the proof follows from this fact and Lemmas 2 and 3. ■

The discrete part of the proposed metric, that captures the length of the (directed) path between two discrete modes, is consistent with the distance notion in graphs: the "discrete ball" or a certain radius k [11]. It would be quite elegant to combine the continuous ball of radius r, B_r, and the discrete ball of radius k, $\mathcal{B}_k$, into a hybrid (open) ball of radius s: B_s^H: $B_s^H \triangleq \{(q, x) \in Q \times X \mid d_H\big((0, 0), (q, x)\big) < s\}$. Such a construction, however, has certain problems. First, the underlying space $Q \times X$ is not a vector space, and the continuity of d_H can only be ensured in certain topologies on Q (other than the discrete one). One can show that under starting from a certain family of open sets on Q, one can define non-empty join irreducibles on X, and discrete open sets on Q, so that the discrete part of d_H can be recast as an AD Nerode-Kohn map [3], which is continuous by construction. Whether the topologies generated by the open sets defined in this process are useful and meaningful for analysis and design, however, is an open issue. For this reason, we will follow the standard route of [9], and explicitly (re)define the stability of Hybrid Automata in the Lyapunov sense in the following section.

V. Hybrid Notions of Stability

Definition 12 (Invariant Set [9]) *A set $W \subseteq Reach_H$ is invariant if $\forall (q[0], x[0]) \in W$, $(\tau, q, x) \in \epsilon_H(q[0], x[0])$, $i \in \langle\tau\rangle$, and $t \in I[i] \Rightarrow (q[i], x[i](t)) \in W$.*

Definition 13 (Stable Invariant Set [9]) *An invariant set W is called*

- *stable if for all $\xi > 0$ there exists a $\delta > 0$ such that for all $(q[0], x[0]) \in Reach_H$ with $d_H((q[0], x[0]), W) < \delta$, all $(\tau, q, x) \in \epsilon_H(q[0], x[0])$, and all $i \in \langle\tau\rangle$, $t \in I[i]$, $d_H((q[i], x[i](t), W)) < \xi$;*
- *L is called asymptotically stable if it is stable and in addition there exists a $\Delta > 0$ such that for all $(q[0], x[0]) \in Reach_H$ with $d_H((q[0], x[0]), W) < \Delta$ and all $(\tau, q, x) \in \epsilon_H^\infty(q[0], x[0])$, $\lim_{t \to |\tau|} d_H((q[i], x[i](t)), W) = 0$.*

Note that positive limit sets L^+ are invariant but not necessarily stable. The existence of L^+ merely suggests that the hybrid trajectory will approach it *in time,* not that it will stay in its neighborhood. We use the positive limit sets to ensure that a transition between discrete modes will occur in finite time. In this paper, stability of a hybrid system H, is understood as convergence to a asymptotically stable invariant set W. For simplicity, we will assume that H has only one (globally) asymptotically stable invariant set W:

Assumption 3 *Assume that the Hybrid Automaton $H = (Q, X, f, \text{Init}, D, E, G, R)$ has only one asymptotically stable invariant set denoted (Q_{eq}, X_{eq}). In addition assume* *that every $q \in Q$ there exists a unique possible discrete jump $e = (q, q') \in E$, and that the associated guard $G(e)$ containing a connected component of $L^+(q)$ is "forced" (the transition must occur).*

VI. Finite Time Abstraction for Continuous Dynamics

We could call two points p_1, p_2 in M equivalent if their positive limit points belong to the same limit set. However, a finer partition of the state space can be achieved by comparing the distances of the flows from two points, p and z, to the same connected component L_k^+ in time T. Having set a time limit on the evolution from p and z, we define a *finite time abstraction*:

Definition 14 (Finite-time Equivalence relations) *Consider an autonomous system (Z, f), where Z is a compact subset of a Banach manifold M, and let the flows of (Z, f) belong in Z for all $t > 0$. Let $L^+ = \bigcup_{k=1}^{\ell} L_k^+$ be the positive limit of (Z, f), where each L_k^+ is simply connected. We define an equivalence relation $\sim_T$ on Z as follows: Two points z_1, $z_2 \in Z$ are said to belong to the same T-equivalence class, and we write $z_1 \sim_T z_2$, if*

1) *$z_1 \sim z_2$, and*
2) *if for some k, $\lim_{t\to\infty} \operatorname{dist}\big(\Phi_t(z_1), L_k^+\big) = \lim_{t\to\infty} \operatorname{dist}\big(\Phi_t(z_2), L_k^+\big) = 0$, then $\operatorname{dist}\big(\Phi_T(z_1), L_k^+\big) = \operatorname{dist}\big(\Phi_T(z_2), L_k^+\big)$.*

The first condition excludes the possibility of one point belonging into different T-equivalence classes. A finite time abstraction partitions the state space according to the distance of the flows of the points at time T, to the component L_k^+ of the positive limit set which they converge to. We use $L_k^+ + \mathcal{B}_d$ to denote the set $\{x + y \mid x \in L_k^+, y \in \mathcal{B}_d\}$.

Definition 15 (Finite Time Abstraction) *Consider a system (Z, f), where Z is a compact subset of a Banach manifold M, and let $\Phi_t(p)$ is the flow of f from $p \in M$. Suppose that the flows of (Z, f) belong in Z for all $t > 0$ and that (Z, f) has a positive limit set $L^+ = \bigcup_{i=1}^{\ell} L_i^+$. The finite-time T-abstraction of (M, f) is a (set valued) map, that associates each point $p \in M$ to the set $L_k^+ + \mathcal{B}_d$, where k is such that $\lim_{t\to\infty} \operatorname{dist}\big(\Phi_t(p), L_k^+\big) = 0$, $d = \operatorname{dist}\big(\Phi_T(p), L_k^+\big)$, and $\mathcal{B}_d$ is the ball of radius d centered at the origin.*

In this sense, a finite-time T-abstraction will retain information about "how close to destination" the flows from different points will be, in time T.

VII. Discrete Asymptotic Abstraction

Consider a Hybrid Automaton H satisfying the conditions of Assumption 2, and let $\Phi_t(q, y)$ be the flow of $f(q, x)$ from $y \in D(q) \setminus G(q, q')$. Given that $L^+(q) \subset G(q, q')$, there will be a (finite) upper bound on the time needed for the flow of $f(q, x)$ to reach $G(q, q')$ from any point $x \in D(q)$. We denote this bound Θ_q. The existence of Θ_q is guaranteed by the definition of the positive limit set $L^+(q)$, and the fact that the latter is completely contained in the guard.

Definition 16 (Finite Time Mode Abstraction) *The Θ_q-abstraction of the dynamical system $(D(q), f(q, x))$ in mode q is given as the image of the constant map: $(M, f(q, \cdot)) \to Q \times \mathbb{R} : (x, f(q, x)) \mapsto (q, \Theta_q)$.*

In this way, the continuous dynamics are dropped completely. All the information that remains is an indication of "how long it takes to reach the guard." This is the concept that allows us to abstract the continuous dynamics of the hybrid system H into a "clock" of a timed automaton.

In abstracting H into a Timed Automaton $\tilde{H}$, we have to consider the equilibrium hybrid set (Q_{eq}, X_{eq}) separately. An ϵ-neighborhood of this set (for an arbitrarily small ϵ,) will give rise to a new "final mode" $\tilde{q}_f$:

$$D(\tilde{q}_f) \triangleq X_{eq} + B_\varepsilon, \quad \varepsilon > 0;$$
$$G(\tilde{q}_f, q') \triangleq D(\tilde{q}_f), \quad \text{implying } q' \equiv \tilde{q}_f;$$
$$R((\tilde{q}_f, \tilde{q}_f), x) \triangleq \text{ identity } , x \in D(\tilde{q}_f).$$

the domain of which coincides with its guard (and therefore $\Theta_q = 0$). Based on Assumption 3, one can obtain the following constructive process for defining the Timed Automaton $\tilde{H}$ that captures the asymptotic behavior of H:

Definition 17 (Abstract timed automaton) *Construct a timed automaton $\tilde{H}$ such that:*

- *$\tilde{Q} = Q \bigcup \{\tilde{q}_f\}$, where $\tilde{q}_f$ is a new mode that represents $X_{eq} \in D(Q_{eq})$.*
- *$\tilde{X}$ is a finite set of continuous variables, where each $\tilde{x} = (\lambda, \gamma)^T$.*
- *$\tilde{f}(\tilde{q}, \lambda, \gamma) = \mathbf{1}$ for all $\tilde{q} \in \tilde{Q}$ (clock dynamics).*
- *$\tilde{Init} \subseteq \tilde{Q} \times \tilde{X}$.*
- *$\tilde{D} : \tilde{Q} \to P(\tilde{X}) | \lambda \leq \Theta_{\tilde{q}}$, $\tilde{q} \in \tilde{Q}$.*
- *$\tilde{E} : \tilde{Q} \times \tilde{Q} \bigcup \{(\tilde{q}_f, \tilde{q}_f)\}$.*
- *$\tilde{G} : \tilde{E} \to P(\tilde{X}) | \lambda \geq \Theta_{\tilde{q}}$, $\tilde{q} \in \tilde{Q}$.*
- *$\tilde{R} : \tilde{E} \times \tilde{X} \to P(\tilde{X}) | (\lambda[i+1], \gamma[i+1])^T = (0, \gamma'[i])^T$.*

where $(\tilde{q}, \tilde{x}) \in \tilde{Q} \times \tilde{X}$ is the state of $\tilde{H}$.

We now present the two main results of this paper. The goal of these two theorems is two study the stability, and the reachability of a hybrid system using an abstract version of it. We do this by abstracting most of the continuous dynamics (Definition 17) of the hybrid automaton keeping only the relevant information to preserve the stability and reachability properties of the system.

Theorem 1 (Asymptotic Stability is preserved) *If H is asymptotically stable (AS) with a S being an ϵ-neigborhood of its AS invariant set (Q_{eq}, X_{eq}), then the timed automaton $\tilde{H}$ constructed as in Definition 17 is asymptotically stable in the sense of Definition 13, with $(\tilde{q}_f, 0)$ its asymptotically stable invariant state.*

Proof: If S is AS then by definition there exists a $\delta > 0$ for all $\xi > 0$, such that for all $(q[0], x[0]) \in Reach_H$

with $d_H((q[0],x[0]),L) < \delta$, every execution $(\tau,q,x) \in \epsilon_H(q[0],x[0])$ will satisfy $d_H((q^{[i]},x^{[i]}(t)),L) < \xi$ for all $i \in \langle\tau\rangle$ and $t \in I[i]$, and there will also exist a $\Delta > 0$ such that for all $(q[0],x[0]) \in Reach_H$ with $d_H((q[0],x[0]),L) < \Delta$ every execution $(\tau,q,x) \in \epsilon_H^{\infty}(q[0],x[0])$ will satisfy $\lim_{t\to|\tau|} d_H((q[i],x[i](t)),L) = 0$. Then by the construction of $\tilde{H}$, and the definition of $d_H(h,h')$ there will exist a $\tilde{\delta} = \lfloor\delta\rfloor + 1$ for all $\tilde{\xi} = \lfloor\xi\rfloor + 1$ (+1 is added due to the addition of $\tilde{q}_f$ in Def. 17) such that for all $(\tilde{q}[0],\tilde{x}[0]) \in Reach_{\tilde{H}}$ with $d_D(\tilde{q}[0],\tilde{q}_f) < \tilde{\delta}$, every execution $(\tilde{\tau},\tilde{q},\tilde{x}) \in \epsilon_{\tilde{H}}(\tilde{q}[0],\tilde{x}[0])$ will satisfy $d_D(\tilde{q}[i],\tilde{q}_f) < \tilde{\xi}$ for all $i \in \langle\tau\rangle$ and $t \in I[i]$ and there will also exist a $\tilde{\Delta} = \lfloor\Delta\rfloor$ such that for all $(\tilde{q}[0],\tilde{x}[0]) \in Reach_{\tilde{H}}$ with $d_D(\tilde{q}[0],\tilde{q}_f) < \tilde{\Delta}$ every execution $(\tilde{\tau},\tilde{q},\tilde{x}) \in \epsilon_{\tilde{H}}^{\infty}(\tilde{q}[0],\tilde{x}[0])$ will satisfy $\lim_{t\to|\tilde{\tau}|} d_D(\tilde{q}[i],\tilde{q}_f) = 0$, thus making $\tilde{q}_f$ the a.s. discrete invariant set of $\tilde{H}$. Since the continuous part of the AS invariant set of $\tilde{H}$ is the whole domain of q_f, the theorem is proved. ■

Let $(q(T),x(T)) \in Reach_H$ denote the state of the hybrid system H at time T. The next Theorem states that the (finite time) reachability properties of H are preserved by $\tilde{H}$:

Theorem 2 *(Reachability of H) If $(q(T),x(T)) \in Reach_H$ there exists a $k \in \mathbb{N}$ such that after some execution $\epsilon_H(q[0],x[0])$, $q(T) = q[k]$, $x(T) = x[k](T)$ with $T \in I[k]$. Moreover T will be upper-bounded by the by $\gamma'[k]$ (the second component of the continuous state of $\tilde{H}$ at the end of mode k), i.e. $T \leq \gamma'[k]$.*

Proof: Let the hybrid state $(q,x) \in Reach_H$ then the abstract hybrid state $\tilde{q},\tilde{x}$ will be in $Reach_{\tilde{H}}$ by Definition 17. Assume that the hybrid automaton starts at the initial conditions $(q[0],x[0](0))$. Then there exists a hybrid execution $\epsilon_H(\tau,q,x)$ that will map the initial condition $(q[0],x[0](0))$ to an state (q,x) such that q will be equal to the discrete state at a $k \in \langle\tau\rangle$ and the corresponding x will be equal to the continuous state at a $k \in \langle\tau\rangle$ and a $T \in I[k]$, i.e. $(q,x) = (q[k],x[k](T))$ such that $k \in \langle\tau\rangle$ and $T \in I[k]$. This state (q,x) is the state of the hybrid system at a time T: $(q,x) = (q(T),x(T))$ along the execution $\epsilon_H(\tau,q,x)$. If a timed automaton $\tilde{H}$ is constructed as in definition 17 $(\lambda,\gamma)^T \in \tilde{X}$ correspond to the local and global clocks of $\tilde{H}$. So by the definition of $\tilde{D}$ and $\tilde{G}$,

$$\lambda'[k] = \Theta_{q[k]} \geq \tau'[k] - \tau[k] \tag{2}$$

By the construction of $\tilde{R}$ in Definition 17 $\gamma'[k] = \sum_{i=1}^{k} \lambda'[i]$. Then using (2) and noting that $T \in [\tau[k],\tau'[k]]$, $\gamma'[k] \geq \sum_{i=1}^{k}(\tau'[i] - \tau[i]) \geq T$. Thus $T \leq \gamma'[k]$. ■

VIII. Conclusions

We define a new distance for hybrid dynamical systems, composed by two completely identifiable parts: a discrete part that is the number of transitions separating two discrete modes, and a continuous part that is a function of a standard distance (induced by a norm) between their corresponding continuous states. Using this distance metric, we introduce the notion of Finite Time Mode Abstraction for a special class of (convergent) hybrid systems. According to this concept, most of the continuous dynamics of the hybrid system is abstracted away, leaving only information about the time that takes a continuous state to reach a transition guard within each particular discrete mode. This information is then used to construct a timed automaton which is shown to preserve the stability and reachability properties of the original hybrid system. Our current analysis applies to the class of hybrid automata with one guard per mode and only one asymptotically stable equilibrium set, but we suggest a procedure for generalization more general classes of hybrid systems, through refinement of their discrete modes. We consider this work as the first step in a path that will allow us to map continuous and hybrid dynamics into (almost completely) discrete ones.

Acknowledgement

The work of the second author is supported by the NSF Career Award #0447898. The work of the first and third authors is supported by NSF under the FIND initiative.

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